# Soft-Collinear Effective Theory

## Part III: Generalization to QCD

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# Outline

Part I: The strategy of regions

Part II: Scalar SCET

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Part IV: Resummation by RG evolution

Part V: IR divergences of gauge theory amplitudes

# Generalization to QCD

The basic structure of the effective theory Lagrangian is the same in QCD as in the scalar theory, but three complications arise

- Different components of the quark and gluon fields $q_\alpha(x)$ and $A_\mu(x)$ scale differently.
- Theory is invariant under gauge transformations, but need to make sure they respect the power counting.
- Non-local operators include Wilson lines to ensure gauge invariance

To make things simpler, let's only consider one type of collinear field with $p_c^\mu \sim (\lambda^2, 1, \lambda)$ for the moment and decompose

$$A^\mu(x) \to A_c^\mu(x) + A_s^\mu(x)$$

$$\psi^\mu(x) \to \psi_c^\mu(x) + \psi_s^\mu(x)$$

Split the fermion field into $\psi_c^\mu(x) = \xi(x) + \eta(x)$

$$\xi = P_+ \psi_c = \frac{\not{n}\not{\bar{n}}}{4}\psi_c \,, \qquad \eta = P_- \psi_c = \frac{\not{\bar{n}}\not{n}}{4}\psi_c$$

So that $\not{n}\,\xi(x) = \not{\bar{n}}\,\eta(x) = 0$ .

P are projection operators:

$$P_+^2 = P_+ \,, \qquad P_-^2 = P_- \,, \qquad P_+ + P_- = 1$$

# Fermion propagator:

$$\langle 0|\mathbf{T}\left[\xi(x)\bar{\xi}(0)\right]|0\rangle = \frac{\not{n}\not{\bar{n}}}{4}\langle 0|\mathbf{T}\left[\psi_c(x)\bar{\psi}_c(0)\right]|0\rangle\frac{\not{\bar{n}}\not{n}}{4}$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ipx}\underbrace{\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4}}_{\frac{\not{n}}{2}\bar{n}\cdot p} \sim \lambda^4\frac{1}{\lambda^2}$$

This implies $\xi(x)\sim\lambda$ . The same argument yields the scaling $\eta(x)\sim\lambda^2$. For the soft quark field

$$\psi_s(x)\bar{\psi}_s(0)\sim(\lambda^2)^4\frac{1}{\lambda^4}\lambda^2\sim\lambda^6$$

so $\psi_s(x)\sim\lambda^3$.

Gluon propagator:

$$\langle 0|\mathbf{T}\left[A_c^\mu(x)A^\nu(0)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ipx}\left[-g_{\mu\nu}+\xi\frac{p_\mu p_\nu}{p^2}\right]$$

Unsurprisingly, the gauge field scales like a derivative, so $A_s^\mu(x) \sim p_s^\mu$ and $A_c^\mu(x) \sim p_c^\mu$, or explicitly

$$n\cdot A_c(x) \sim \lambda^2 \quad \bar{n}\cdot A_c(x) \sim \lambda^0 \quad A_{c\perp}^\mu(x) \sim \lambda$$

The soft gluon field is power suppressed compared to the collinear field, except for the $n\cdot A_s$ component.

The collinear quark Lagrangian has a special form. Since the $\eta(x)$ components of the field are of higher order than the $\xi(x)$ components, and using $\not{n}\,\xi(x) = \not{\bar{n}}\,\eta(x) = 0$, we decompose

$$\mathcal{L}_c = \bar{\psi}_c\, i\not{D}_c\, \psi_c = (\bar{\xi} + \bar{\eta}) \left[ \frac{\not{n}}{2} i\bar{n}\cdot D + \frac{\not{\bar{n}}}{2} in\cdot D + i\not{D}_\perp \right] (\xi + \eta)$$

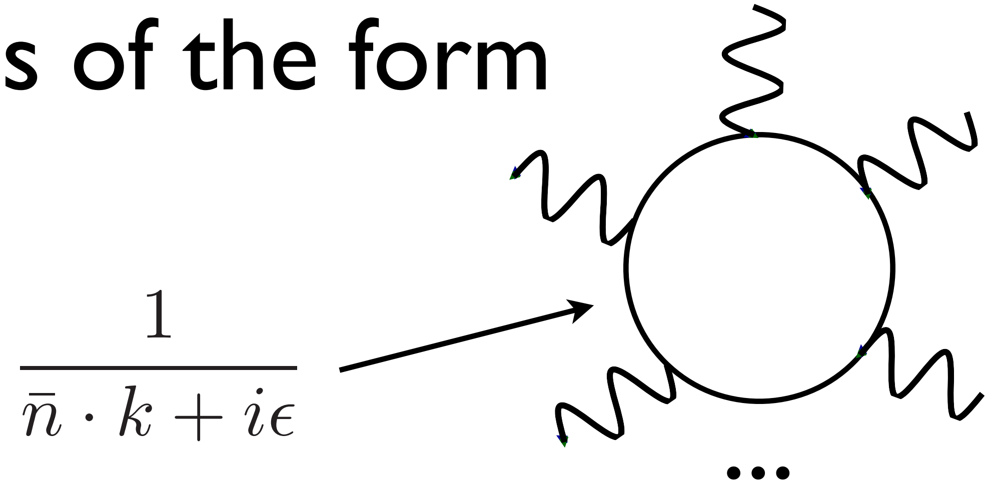
$$= \bar{\xi}\, \frac{\not{\bar{n}}}{2} in\cdot D\, \xi + \bar{\xi}\, i\not{D}_\perp\, \eta + \bar{\eta}\, i\not{D}_\perp\, \xi + \bar{\eta}\, \frac{\not{n}}{2} i\bar{n}\cdot D\, \eta.$$

To get a simple power counting, we integrate out the $\eta(x)$ components. This can be done exactly because the action is quadratic.

One obtains (*exercise*)

$$\mathcal{L}_c = \bar{\xi}\left[\frac{\bar{\not n}}{2} i\bar{n}\cdot D - i\not{D}_\perp \frac{\bar{\not n}}{2}\frac{1}{i\bar{n}\cdot D} i\not{D}_\perp\right]\xi$$

In addition, integrating out the $\eta(x)$ components gives an irrelevant determinant $\det\left(\frac{\not n}{2} i\bar{n}\cdot D\right)$. To see that it amounts to a trivial overall factor, note that it is gauge invariant and *A*$_\mu$ independent in $\bar{n}\cdot A = 0$ gauge. The determinant corresponds to diagrams of the form



$$\frac{1}{\bar{n}\cdot k + i\epsilon}$$

vanish, only poles at $\mathrm{Im}\left[\,\bar{n}\cdot k\,\right] < 0$

While the collinear quark Lagrangian has a somewhat complicated form, the collinear gluon Lagrangian is just the QCD expression with $A^\mu$ replaced by $A_c^\mu$

The same is true for the soft Lagrangian

$$\mathcal{L}_s = \bar{q}_s \, i \not{D}_s \, q_s - \frac{1}{4} (F_{\mu\nu}^{s\,a})^2$$

with $iD_s^\mu = i\partial^\mu + gA_s^\mu$ and $igF_{\mu\nu}^{s\,a} \, t^a = [iD_s^\mu, iD_s^\mu]$

We end up with two copies of the QCD Lagrangian (one with an uncommon form of the quark part). Only the soft-collinear interactions are missing.

# Soft-collinear interactions

The general construction (see Beneke and Feldmann hep-ph/0211358) of the interactions is somewhat involved. For collider physics applications the leading power soft-collinear interactions are usually sufficient.

- Since $q_s$ is power suppressed with respect to $\xi$, no soft quarks appear in leading power interactions.
- Only the $n{\cdot}A_s$ component can appear, all other components are suppressed compared to their collinear counterparts.

The soft-collinear interactions are obtained from the collinear Lagrangian by making the substitution

$$A_c^\mu(x) \longrightarrow A_c^\mu(x) + n \cdot A_s(x_-)\frac{\bar{n}^\mu}{2}$$

# Leading-power SCET Lagrangian

$$\mathcal{L}_{\mathrm{SCET}} = \bar{\xi}\,\frac{\bar{\not{n}}}{2}\left[in\cdot D + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c}i\not{D}_{c\perp}\right]\xi - \frac{1}{4}(F^{c\,a}_{\mu\nu})^2 + \bar{q}_s\, i\not{D}_s\, q_s - \frac{1}{4}(F^{s\,a}_{\mu\nu})^2$$

where

$igF^{s\,a}_{\mu\nu}\;t^a = [iD^\mu_s, iD^\mu_s]$ and $igF^{c\,a}_{\mu\nu}\;t^a = [iD^\mu, iD^\mu]$

with

$$iD^\mu_s = i\partial^\mu + g\,A_s(x)^\mu$$

$$iD^\mu_c = i\partial^\mu + g\,A_c(x)^\mu$$

$$iD^\mu = i\partial^\mu + g\,A^\mu_c(x) + g\,n\cdot A_s(x_-)\frac{\bar{n}^\mu}{2}$$

As in the scalar theory, this Lagrangian is exact, i.e. there are no matching corrections.

# Gauge invariance

When performing gauge transformations, we must make sure that they respect the scaling of the fields. For example, transforming a soft field under a gauge transformation $\alpha(x)$ with collinear scaling would turn it into collinear field.

We consider two gauge transformations

$$V_s(x) = \exp\left[\, i\alpha_s^a(x)\, t^a \,\right] \qquad V_c(x) = \exp\left[\, i\alpha_c^a(x)\, t^a \,\right]$$

where $\alpha_{\rm s}(x)$ has soft scaling, i.e. $\partial^\mu \alpha_s(x) \sim \lambda^2 \alpha_s(x)$, and $\alpha_{\rm c}(x)$ collinear scaling.

# Soft gauge transformations

The soft fields transform in the usual way

$$\psi_s(x) \to V_s(x)\,\psi_s(x) \qquad A_s^\mu(x) \to V_s(x) A_s^\mu(x) V_s^\dagger(x) + \frac{i}{g} V_s(x)\left[\partial^\mu, V_s^\dagger\right]$$

while the collinear fields transform as

$$\xi(x) \to V_s(x_-)\,\xi(x) \qquad A_c^\mu(x) \to V_s(x_-) A_c^\mu(x) V_s^\dagger(x_-)$$

The $x_-$ instead of $x$ dependence ensures that the transformation does not induce higher power corrections. Also, the $V_s(x)\left[\partial^\mu, V_s^\dagger\right]$ is a higher power correction for $A_{c\perp}^\mu$ and $\bar{n}\cdot A_c$. The small component $n\cdot A_c$ only appears in the combination

$$in\cdot D \to V_s(x_-)\, in\cdot D\, V_s^\dagger(x_-)$$

# Collinear gauge transformation

Under collinear transformations, the soft fields remain unchanged. The collinear fields transform as

$$\xi(x) \to V_c(x)\,\xi(x)\,, \quad A_c^\mu(x) \to V_c(x) A_c^\mu(x) V_c^\dagger(x) + \frac{1}{g} V_c(x) \left[ i\partial^\mu + \frac{\bar{n}^\mu}{2} n \cdot A_s(x_-), V_c^\dagger(x) \right]$$

The soft piece in this transformation law ensures that

$$in \cdot D \to V_c(x)\, in \cdot D\, V_c^\dagger(x).$$

Both collinear and soft gauge transformations are homogenous and leave the Lagrangian invariant (*exercise*). A thorough discussion of the gauge transformations and the construction of higher power terms in the Lagrangian is given in Beneke and Feldmann hep-ph/0211358.

# Wilson lines and decoupling transformation

In the scalar theory, we encountered operators such as

$$J_2(x) = \int ds\, dt\, C_2(s,t)\, \phi_{c1}(x+s\bar{n})\phi_{c2}(x+tn)$$

In a gauge theory, products of fields at different points are made gauge invariant by connecting them with Wilson lines, such as

$$[x+s\bar{n}\,,\ x] \equiv \mathbf{P}\exp\left[ig\int_0^s ds'\,\bar{n}\cdot A(x+s'\bar{n})\right]$$

$\mathbf{P}$ indicates path ordering of the color matrices. Under gauge transformations

$$[x+s\bar{n}\,,\ x] \to V(x+s\bar{n})\ [x+s\bar{n}\,,\ x]\,V^\dagger(x)$$

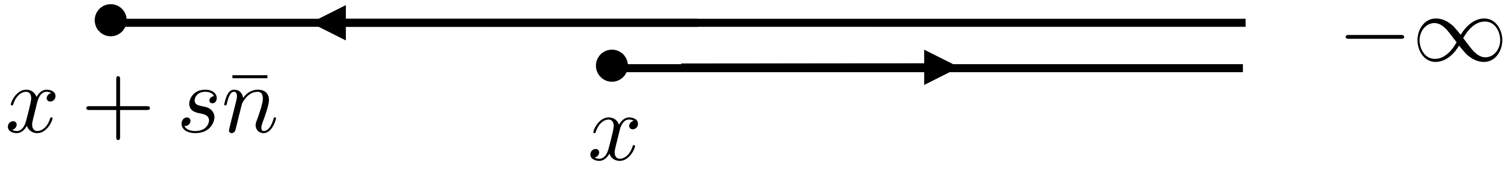
so $\psi(x+s\bar{n})\,[x+s\bar{n}\,,\ x]\,\psi(x)$ stays invariant.

Instead of working with Wilson lines [ $x$+ $s\,\bar{n}$, $x$ ], it is customary in SCET to work with

$$W(x) = \mathbf{P}\exp\left[ig\int_{-\infty}^{0} ds\,\bar{n}\cdot A(x+s\bar{n})\right] = [x\,,\,-\infty\,\bar{n}\,]$$

The finite segment is obtained from the product

$$[x+s\bar{n}\,,\;x] = W(x+s\bar{n})\,W^\dagger(x)$$



The combination $W^\dagger(x)\psi(x)$ is invariant under gauge transformations which vanish at infinity.

The covariant derivative along the Wilson line vanishes:

$$\bar{n} \cdot D\, W(x) = 0$$

It parallel transports the phase of the gauge field.

Two types of Wilson lines are used in SCET:

$$W_c(x) = \mathbf{P} \exp \left[ ig \int_{-\infty}^{0} ds\, \bar{n} \cdot A_c(x + s\bar{n}) \right] \,, \qquad \text{(collinear)}$$

$$S_n(x) = \mathbf{P} \exp \left[ ig \int_{-\infty}^{0} ds\, n \cdot A_s(x + sn) \right] \qquad \text{(soft)}$$

The collinear line is useful to construct operators, while the soft line is useful because of the structure of the soft interaction.

# Decoupling transformation

The unitary field redefinition

$$\xi(x) \to S_n(x_-)\xi^{(0)}(x)\,,$$

$$A_c^\mu(x) \to S_n(x_-)\,A_c^{(0)\,\mu}(x)\,S_n^\dagger(x_-)$$

decouples soft and collinear interactions:

$$\mathcal{L}_{\rm int} = \bar\xi\,\frac{\not{n}}{2} i\bar{n}\cdot D\,\xi$$

$$\to \bar\xi^{(0)}\,S_n^\dagger(x_-)\,\frac{\not{n}}{2}\left[i\bar{n}\cdot\partial + gS_n(x_-)\,\bar{n}\cdot A_c^{(0)}\,S_n^\dagger(x_-) + g\bar{n}\cdot A_s(x_-)\right]S_n(x_-)\xi^{(0)}$$

$$= \bar\xi^{(0)}\,\frac{\not{n}}{2}\,\bar{n}\cdot A_c^{(0)}\xi^{(0)} + \bar\xi^{(0)}\,S_n^\dagger(x_-)\,\frac{\not{n}}{2}\,n\cdot D_s S_n(x_-)\xi^{(0)}$$

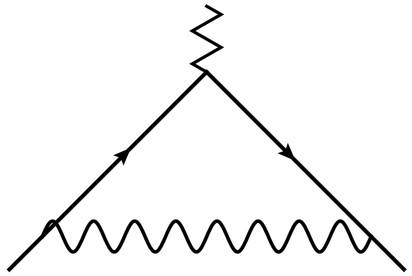
$$= \bar\xi^{(0)}\,\frac{\not{n}}{2}\,\bar{n}\cdot A_c^{(0)}\xi^{(0)} + \bar\xi^{(0)}\,\frac{\not{n}}{2}\,n\cdot\partial\,\xi^{(0)}(x) = \underline{\bar\xi^{(0)}\,\frac{\not{n}}{2}\,n\cdot D_c\,\xi^{(0)}}$$

# Decoupling transformation

Note that the field redefinition only decouples the leading power Lagrangian. At subleading power, soft-collinear interactions remain.

More importantly, the decoupling does not imply that everything factorizes at leading power. For example, to analyze the Sudakov problem, we need to match the vector current operator. And while the soft interactions decouple from the Lagrangian, they remain in the current operator, as we'll see now.

# Sudakov problem in QCD



To analyze the Sudakov problem in QCD, we need to match the vector current onto SCET. The leading power expression is

$$\psi(x)\gamma^\mu\psi(x) \to \int dsdt\, C_V(s,t)\, \chi_{c1}(x+s\bar{n})\, \gamma_\perp^\mu\, \chi_{c2}(x+tn)$$

with two collinear quark fields

$$\chi_{c1}(x) = W_{c1}^\dagger(x)\, \xi_{c1}(x)\,, \qquad \chi_{c2}(x) = W_{c2}^\dagger(x)\, \xi_{c2}(x)$$

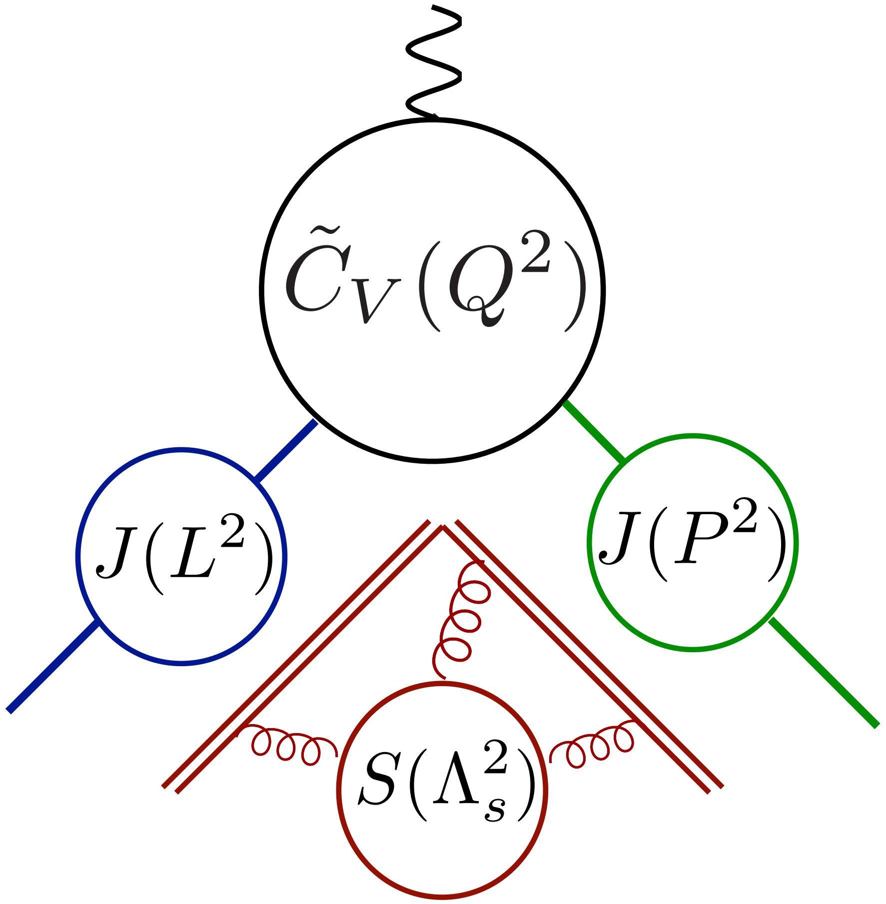
which fulfill $\not{n}\, \chi_{c1}(x) = \not{\bar{n}}\, \chi_{c2}(x) = 0$.

Performing the decoupling transformation, we get:

$$\chi_{c1}(x+s\bar{n})\,\gamma_\perp^\mu\,\chi_{c2}(x+tn)$$

$$\to\ \chi_{c1}^{(0)}(x+s\bar{n})\,S_n^\dagger(x_-)\,\gamma_\perp^\mu\,S_{\bar{n}}(x_+)\,\chi_{c2}^{(0)}(x+tn)$$

The soft interactions do *not* cancel.



$$\Lambda_s^2 = \frac{L^2P^2}{Q^2}$$

# Factorization vs. factorization

Unfortunately, the word factorization is used in two different meanings.

1. Factorization = scale separation. We have separated the pieces associated with the different scales in the problem, so in this first sense we have factorized the form factor.

2. Factorization = no low energy interactions. Since there are soft interactions, the two collinear sectors interact. The form factor is not factorizable in this sense. (The total cross section at a hadron collider, for example, factorizes also in the second sense.)