Problem Set 2.

Quasiclassical limit and some algebra.

Problem 2.1 A little bit of algebra.
Using the Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \tag{1}$$

prove the operator series

$$(A-B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \dots \tag{2}$$

where A and B are linear operators (generally non-commuting).

This series is convergent for sufficiently small B (can you specify the condition of convergence?)

Problem 2.2 Gaussian integrals.

Integrals of exponents of quadratic polynomials are called "gaussian". The simplest such integral is

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \tag{3}$$

(1) Consider a symmetric matrix A_{ij} of dimension N and prove that

$$Z \equiv \int_{-\infty}^{+\infty} dx_1 \dots dx_N \, \exp\left(\sum_{ij} x_i A_{ij} x_j\right) = \frac{\pi^{N/2}}{\sqrt{\det A}} \tag{4}$$

Hint: do the integration in the basis of eigenvectors of A.

(2) Compute the average

$$\langle x_k x_l \rangle = \frac{1}{Z} \int_{-\infty}^{+\infty} dx_1 \dots dx_N \, x_k x_l \, \exp\left(\sum_{ij} x_i A_{ij} x_j\right) \tag{5}$$

Problem 2.3

Consider the propagator of the free one-dimensional particle

$$K(t_f, x_f; t_i, x_i) = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \exp\left[\frac{i}{\hbar}\frac{M(x_f - x_i)^2}{2(t_f - t_i)}\right] \tag{6}$$

To estimate how much the quantum particle deviates from its classical trajectory, we can calculate the following "average deviation"

$$\langle \delta x^2(t) \rangle \equiv \frac{\int dx \, K(t_f, x_f; t, x) \, x^2 \, K(t, x; t_i, x_i)}{K(t_f, x_f; t_i, x_i)} \tag{7}$$

Compute this "average". Why is it not real?

Problem 2.4

As we have shown in class, the quasiclassical expression for the propagator is

$$K(t_f, x_f; t_i, x_i) = \sqrt{\frac{m}{2\pi i\hbar\varphi(t_f)}} \exp\left[\frac{i}{\hbar}S_{cl}\right] \tag{8}$$

where $\varphi(t)$ is a function defined as the solution to the differential equation

$$m\ddot{\varphi} + U''(x_{cl}(t))\varphi = 0 \tag{9}$$

with the boundary conditions

$$\varphi(t_i) = 0\,, \qquad \dot{\varphi}(t_i) = 1 \tag{10}$$

We want to reproduce the same result with the quasiclassical wave functions in the Hamiltonian formalism (with a time-independent Hamiltonian). We use

$$K(t_f, x_f; t_i, x_i) = \int \frac{dE}{2\pi} \Psi_E(x_f)\Psi_E^*(x_i)\, e^{-iE(t_f - t_i)} \tag{11}$$

where Ψ_E are the eigenfunctions at energies E. The wave functions in this formula must be normalized so that

$$\int dx \Psi_E(x)\Psi_{E'}^*(x) = 2\pi\delta(E - E') \tag{12}$$

The quasiclassical expression with the appropriate normalization is

$$\Psi_E(x) = \sqrt{\frac{m}{\hbar p(x)}} \exp\left[\frac{i}{\hbar}\int^x p(x')\, dx'\right]\,, \tag{13}$$

where $p(x) = \sqrt{2m(E - U(x))}$.

Substitute this expression into (11), expand to the second order in fluctuations around the classical action, and obtain the propagator in the form (8) with

$$\varphi(t) = mp(x)p(x_i)\int_{x_i}^{x} \frac{dx'}{p^3(x')} \tag{14}$$

where $x = x_{cl}(t)$ is the classical trajectory of the particle. Finally, show that this $\varphi(t)$ obeys the equations (9), (10).