Problem Set 4.

Path integrals in imaginary time. Motion in a magnetic field.

Problem 4.1. Return probability in a random walk.

Consider a particle jumping randomly on a D-dimensional qubic lattice. At time $t = 0$ it starts from the site 0, and at each time step (time is integer in this problem) it jumps with equal probability to any of the $2D$ adjacent sites. What is the probability P_0 that the particle never returns to the initial site? Show that this probability is 0 in dimensions $D \leq 2$ and finite in $D \geq 3$.

Hints: Denote $p(x,t)$ the probabilty to find the particle at position x at time t (x is a D-dimensional vector). Find its Fourier transform $p(k,\omega)$. Next, you may consider the "modified" Green's function $\tilde{p}(x,t)$ for the motion of the particle which never returns to the initial site 0 (if the particle reaches 0, it disappears). This function (and the corresponding equation relating $\tilde{p}(x,t)$ to $p(x,t)$) is similar to the perturbative treatment of a particle in a delta-potential. The probability of never returning to point 0 is $P_0 = \tilde{p}(k=0, t \to \infty)$, which translates into the final result

$$P_0 = [p(x=0, \omega=0)]^{-1} \tag{1}$$

Problem 4.2. Fermions.

Consider N noninteracting fermions at a finite temperature T. The partition function $Z = \operatorname{Tr} \exp(-H/T)$ may be expressed as a path integral over periodic trajectories of fermions in imaginary time (with the period $1/T$). There are different types of such trajectories corresponding to different permutations of particles. The corresponding contributions to the partition function are multiplied by minus one for odd permutations. Show that for $N = 3$, the partition function will be given by

$$Z = \sum_{i \neq j \neq k \neq i} e^{-(E_i + E_j + E_k)/T} \tag{2}$$

(we have performed the argument for $N = 2$ at the lecture).

Can you generalize this result to arbitrary N?

Problem 4.3. Persistent current.

Consider a one-dimensional particle on a ring with the Hamiltonian

$$H = \frac{p^2}{2m} + U(x) \tag{3}$$

where x is a periodic variable ($x + L = x$). The ring is pierced by the magnetic flux Φ. The ground state energy depends periodically on Φ. Show that the current in the ground state may be expressed as

$$I = \frac{\partial E}{\partial \Phi} \tag{4}$$

For the case of a free particle ($U = 0$) plot the energy and the current as a function of Φ.

Problem 4.4. Lorentz force.
The effect of the magnetic field may be included in the action of a charged particle by

$$S = S_0 - e \int \mathbf{A} d\mathbf{x} \tag{5}$$

Show that in the classical limit this term in the action corresponds to the Lorentz force

$$F = e\mathbf{v} \times \mathbf{B} \tag{6}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$.