Problem Set 5.

Landau levels. Spin tunneling by path integral.

Problem 5.1 Landau levels on a torus.

Consider a charged spinless two-dimensional particle of mass m in a box $L_x \times L_y$ with periodic boundary conditions (i.e. particle on a torus). We can impose "magnetic-flux" boundary conditions:

$$\Psi(x + L_x, y) = e^{i\phi_x}\Psi(x, y), \qquad \Psi(x, y + L_y) = e^{i\phi_y}\Psi(x, y) \tag{1}$$

The spectrum obviously depends on the twists ϕ_x and ϕ_y.

Now apply a uniform magnetic field B through the plane.

(a) Show that the problem is well-defined only if the total flux is a multiple of $2\pi/e$ (e is the particle charge):

$$eBL_xL_y = 2\pi N \tag{2}$$

For this problem, define appropriate boundary conditions on the wave function Ψ. Note that for $N \neq 0$ the magnetic vector potential is not single valued: you need to define covering of the torus by two domains (on each of them the vector potential is single-valued), and the wave functions on the overlapping parts of the domains will be identified by a gauge transformation.

(b) Show that with the magnetic field, the spectrum of the particle does not depend on the additional twists ϕ_x and ϕ_y in the boundary conditions. Hint: consider shifts of the wave function in the x and y directions.

(c) Find the spectrum of the particle. You should recover exactly degenerate Landau levels with the degenracy N each. What are the wave functions of those states?

Problem 5.2 Spin tunneling by path integral.

Consider the spin Hamiltonian

$$H = K_x S_x^2 + K_y S_y^2, \qquad K_x > K_y > 0 \tag{3}$$

In the large-spin limit, the tunneling between the two classical minima $S_z = \pm S$ may be studied with the path-integral approach [see D.Loss, D.P.DiVincenzo, G.Grinstein, Phys. Rev. Lett. **69**, 3232 (1992)]. For half-integer spin the tunneling amplitude is zero (destructive interference).

In the case of integer spin S, the tunneling amplitude contains a quasiclassical tunneling action S_{cl}. Calculate this action. The result is

$$S_{cl} = S \ln \frac{1 + \sqrt{\kappa}}{1 - \sqrt{\kappa}}, \qquad \kappa = \frac{K_y}{K_x} \tag{4}$$

Proceed along the following lines:

(1) First, show that the corresponding classical Hamiltonian is

$$\langle \mathbf{n}|H|\mathbf{n}\rangle = \text{const} + S(S-\frac{1}{2})(K_x n_x^2 + K_y n_y^2) \tag{5}$$

(2) Second, find the tunneling trajectory. It should satisfy the system of equations

$$\begin{aligned} n_x^2 + n_y^2 + n_z^2 &= 1 \\ K_x n_x^2 + K_y n_y^2 &= 0 \end{aligned} \tag{6}$$

Obviously, some of the coordinates n_i should become complex. The corresponding Berry phase also becomes complex – and its imaginary part is exactly the "tunneling action" we are looking for.

(3) Finally, find the classical action – the imaginary part of the Berry phase. We continuously connect the complex tunneling trajectory $\mathbf{n}(\theta)$ to an arbitrary real trajectory on the sphere $\mathbf{n}_0(\theta)$. The set of all intermediate trajectories defines a two-dimensional surface Σ parametrized by the two coordinates $\mathbf{n}(\varphi, \theta)$. The *imaginary* part of the Berry phase is then

$$S_{cl} = S\,\mathrm{Im} \int_\Sigma d\varphi\, d\theta \left(\mathbf{n}, \left[\frac{\partial \mathbf{n}}{\partial \phi}, \frac{\partial \mathbf{n}}{\partial \theta}\right]\right) \tag{7}$$

After you do the integral, you should recover the result (4).

P.S. The preprint version of the above mentioned PRL paper is available at `http://xxx.lanl.gov/abs/cond-mat/9208012`.