Problem Set 6.

Bosons and fermions.

Problem 6.1 Coherent states of bosons.
For the coherent state of bosons

$$|a\rangle = e^{-\frac{|a|^2}{2}} \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} |n\rangle \tag{1}$$

(1) Prove that

$$\langle a|d|a\rangle = \frac{1}{2}(a^* \, da - a \, da^*) \tag{2}$$

(d denotes differentiation)

(2) For a closed contour Γ in the complex plane (a, a^*), prove that the integral

$$\int_\Gamma \langle a|d|a\rangle \tag{3}$$

is proportional to the area inside Γ.

Problem 6.2 Consider the ground state of one-dimensional spinless non-interacting fermions. The density of fermions is n.

(1) Find the equal-time Green's function in the gound state

$$\langle c^\dagger(x) c(y)\rangle \tag{4}$$

($c^\dagger$ and c are the creation and annihilation operators for the fermions). Note that it does not depend on the particle mass.

(2) Find the equal-time density-density correlation in the ground state

$$\langle \rho(x)\rho(y)\rangle, \qquad \text{where} \quad \rho(x) = c^\dagger(x) c(x) \tag{5}$$

(3) Consider a large interval of length L. Define the number of particles in this interval as

$$N = \int_0^L \rho(x)\, dx \tag{6}$$

On average, the particle number in this interval is $\langle N\rangle = nL$.

Calculate the quantum fluctuations of N in the ground state

$$(\delta N)^2 = \langle N^2\rangle - \langle N\rangle^2 \tag{7}$$

Show that at large N the fluctuation $(\delta N)^2$ grows as $\ln N$. This contrasts the situation with the gas of classical particles, where the distribution of N is Poissonian with $(\delta N)^2 \sim N$.