Problem Set 3.

Instantons and fluctuation determinants.

Problem 3.1 Instantons and tunneling matrix elements.

Consider N identical potential wells connected by quasiclassical tunneling processes. Let one-instanton contribution to the propagator $K(a_i, \tau^*; a_j, 0)$ connecting two wells a_i and a_j be $K_0(\tau^*) t_{ij} \tau^*$, where $K_0(\tau^*)$ is the oscillator propagator in one well, and t_{ij} are the instanton amplitudes.

Show that the multi-instanton propagator $K(a_i, \tau^*; a_j, 0)$ is then given at large τ^* by

$$K(a_i, \tau^*; a_j, 0) = K_0(\tau^*) \left(e^{-H\tau^*} \right)_{ij}, \tag{1}$$

where H is the $N \times N$ matrix with the elements $H_{ij} = -t_{ij}$. Show further that the splitting of the ground-state energy levels in the wells is given by the eigenvalues of H.

Problem 3.2

Consider the Hamiltonian

$$H_\alpha = -\frac{d^2}{dx^2} - \alpha\delta(x) \tag{2}$$

acting on wave functions $\Psi(x)$ on the interval $x \in [-a, a]$ with the boundary conditions $\Psi(-a) = \Psi(a) = 0$.

(a) Compute $(\det H_\alpha)/(\det H_0)$.

(b) At a certain value $\alpha = \alpha_0$, $\det H_{\alpha_0} = 0$. Find the ground state energy $E_0(\alpha)$ and use it to compute $(\det' H_{\alpha_0})/(\det H_0)$, where $\det'$ is the determinant with the zero mode excluded:

$$\det' H_{\alpha_0} = \lim_{\alpha \to \alpha_0} \frac{\det H_\alpha}{E_0(\alpha)} \tag{3}$$

(c) Verify that the result is consistent with the formula which we derived in class:

$$\frac{\det' H_{\alpha_0}}{\det H_0} = -\frac{1}{2a\Psi_0'(-a)\Psi_0'(a)}, \tag{4}$$

where Ψ_0 is the zero mode of H_{α_0} normalized as $\int_{-a}^{a} |\Psi_0(x)|^2 dx = 1$.

Problem 3.3

For the double-well potential

$$U(x) = \alpha(x^2 - x_0^2)^2 \tag{5}$$

the instanton and the fluctuation determinant may be computed analytically.

Perform this calculation and find the splitting of the two lowest levels, including the pre-exponential factors [the easiest way to calculate the fluctuation determinant is to use the trick (4)]. You may compare the instanton result to the calculation in the Hamiltonian approach (see, e.g., Landau & Lifshits' textbook).

Hints and answers:

1. The instanton has the form

$$x_{cl}(\tau) = x_0 \tanh\left(\frac{\omega}{2}\tau\right). \tag{6}$$

2. The final result for the splitting (half the energy difference between the levels) will be

$$\Delta = \sqrt{\frac{6}{\pi}}\,\omega\,\sqrt{S_{cl}}\,\exp(-S_{cl}) \tag{7}$$

3. The result for the splitting of the ground states in the Hamiltonian approach may be obtained by re-computing the numerical prefactor in the problem in the Landau & Lifshits book (at the end of section 50). The problem in the book deals with the splitting of quasiclassical levels. For the ground state, the pre-exponential factor will be different:

$$\Delta = \frac{\omega}{2\sqrt{\pi e}} \exp\left(-\int_{-a}^{a} |p|\,dx\right), \tag{8}$$

where the integral is taken between the turning points $\pm a$ of the classical trajectory, not between the minima of the wells as in (7). It is not difficult to see that the difference between the definitions of the actions in (7) and (8) is compensated by the $\sqrt{S_{cl}}$ pre-factor in (7), to the logarithmic precision. Verifying that the numerical prefactors in the two answers agree requires a more thourough calculation (you may try!).