

**Problem Set 5.**

Landau levels. Spin tunneling by path integral.

**Problem 5.1** Landau levels on a torus.

Consider a charged spinless two-dimensional particle of mass  $m$  in a box  $L_x \times L_y$  with periodic boundary conditions (i.e. particle on a torus). We can impose “magnetic-flux” boundary conditions:

$$\Psi(x + L_x, y) = e^{i\phi_x} \Psi(x, y), \quad \Psi(x, y + L_y) = e^{i\phi_y} \Psi(x, y) \quad (1)$$

The spectrum obviously depends on the twists  $\phi_x$  and  $\phi_y$ .

Now apply a uniform magnetic field  $B$  through the plane.

(a) Show that the problem is well-defined only if the total flux is a multiple of  $2\pi/e$  ( $e$  is the particle charge):

$$eBL_xL_y = 2\pi N \quad (2)$$

For this problem, define appropriate boundary conditions on the wave function  $\Psi$ . Note that for  $N \neq 0$  the magnetic vector potential is not single valued: you need to define covering of the torus by two domains (on each of them the vector potential is single-valued), and the wave functions on the overlapping parts of the domains will be identified by a gauge transformation.

(b) Show that with the magnetic field, the spectrum of the particle does not depend on the additional twists  $\phi_x$  and  $\phi_y$  in the boundary conditions. Hint: consider shifts of the wave function in the  $x$  and  $y$  directions.

(c) Find the spectrum of the particle. You should recover exactly degenerate Landau levels with the degeneracy  $N$  each. What are the wave functions of those states?

**Problem 5.2** Spin tunneling by path integral.

Consider the spin Hamiltonian

$$H = K_x S_x^2 + K_y S_y^2, \quad K_x > K_y > 0 \quad (3)$$

In the large-spin limit, the tunneling between the two classical minima  $S_z = \pm S$  may be studied with the path-integral approach [see D.Loss, D.P.DiVincenzo, G.Grinstein, Phys. Rev. Lett. **69**, 3232 (1992)]. For half-integer spin the tunneling amplitude is zero (destructive interference).

In the case of integer spin  $S$ , the tunneling amplitude contains a quasiclassical tunneling action  $S_{cl}$ . Calculate this action. The result is

$$S_{cl} = S \ln \frac{1 + \sqrt{\kappa}}{1 - \sqrt{\kappa}}, \quad \kappa = \frac{K_y}{K_x} \quad (4)$$

Proceed along the following lines:

(1) First, show that the corresponding classical Hamiltonian is

$$\langle \mathbf{n} | H | \mathbf{n} \rangle = \text{const} + S(S - \frac{1}{2})(K_x n_x^2 + K_y n_y^2) \quad (5)$$

(2) Second, find the tunneling trajectory. It should satisfy the system of equations

$$\begin{aligned} n_x^2 + n_y^2 + n_z^2 &= 1 \\ K_x n_x^2 + K_y n_y^2 &= 0 \end{aligned} \quad (6)$$

Obviously, some of the coordinates  $n_i$  should become complex. The corresponding Berry phase also becomes complex – and its imaginary part is exactly the “tunneling action” we are looking for.

(3) Finally, find the classical action – the imaginary part of the Berry phase. We continuously connect the complex tunneling trajectory  $\mathbf{n}(\theta)$  to an arbitrary real trajectory on the sphere  $\mathbf{n}_0(\theta)$ . The set of all intermediate trajectories defines a two-dimensional surface  $\Sigma$  parametrized by the two coordinates  $\mathbf{n}(\varphi, \theta)$ . The *imaginary* part of the Berry phase is then

$$S_{cl} = S \text{Im} \int_{\Sigma} d\varphi d\theta \left( \mathbf{n}, \left[ \frac{\partial \mathbf{n}}{\partial \varphi}, \frac{\partial \mathbf{n}}{\partial \theta} \right] \right) \quad (7)$$

After you do the integral, you should recover the result (4).

P.S. The preprint version of the PRL paper is available at <http://xxx.lanl.gov/abs/cond-mat/9208012>.