

# Boundary conformal field theory and D-branes\*

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## Abstract

An introduction to boundary conformal field theory is given with particular emphasis on applications to the construction of D-branes in string theory.

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# 1 Introduction

The past few years have seen a tremendous increase in our understanding of the dynamics of superstring theory. In particular it has become apparent that the five ten-dimensional theories, together with an eleven-dimensional theory (M-theory), are different limits in moduli space of some unifying description. A crucial ingredient in understanding the relation between the different perturbative descriptions has been the realisation that the *solitonic* objects that define the relevant degrees of freedom at strong coupling are *Dirichlet-branes* that have an alternative description in terms of open string theory [1, 2, 3].

D-branes in string theory can be described and analysed in essentially two different ways. First, one can think of D-branes as being extended objects in space-time that can wrap around certain cycles in the target space geometry. From this point of view, D-branes are described by geometrical data such as cohomology and K-theory [4, 5, 6]. On the other hand, as was realised by Polchinski [3], these extended objects can be characterised by their property that open strings can end on them. This is to say, we can describe D-branes in terms of the boundary conditions they impose at the end-points of the open strings. From this point of view, the different D-branes of the theory then simply corresponds to the different open string sectors that can be added consistently to a given (closed) string theory. In terms of the ‘world-sheet’ approach, D-branes are therefore described by (boundary) conformal field theory.

The boundary conformal field theory description is an *exact* string theory description, but it is often only available at specific points in the moduli space of target space geometries, such as orbifold points [7, 8, 9, 56, 11, 12, 13, 14, 15], Gepner points in Calabi-Yau manifolds [16, 17, 18], *etc.* On the other hand, the geometrical approach is generically available, but it can only be trusted whenever we are in a regime where the supergravity approximation is good. The two approaches are therefore in some sense complementary, and one can learn interesting features about ‘stringy geometry’ by comparing their results (see for example [19]).

In these lectures I shall attempt to give a pedagogical introduction to the conformal field theory approach. In section 2 I shall begin by describing some aspects of boundary conformal field theory. I shall mainly consider the situation where the underlying conformal field theory is rational, and the D-branes preserve the full symmetry algebra. In section 3 I shall then describe the simplest application to string theory, the construction of D-branes in the bosonic string theory, as well as the superstring. I shall briefly describe how BPS and non-BPS D-branes can be described from this point of view, and why non-BPS branes may be stable in orbifold theories. I shall furthermore give a very brief outline of how D-brane charges can be described by K-theory. Finally, in

section 4<sup>1</sup> I shall describe more recent work (mainly in collaboration with Michael Green [21, 22, 23]) in which D-branes have been constructed for the maximally supersymmetric plane-wave background. The relevant world-sheet theory in this case is not conformally invariant, but the analysis is nevertheless rather similar.

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<sup>1</sup>This material was recently reviewed in [20] and is therefore not included in this set of notes.

## 2 The boundary conformal field theory approach

Suppose we are given a conformal field theory defined on closed Riemann surfaces, *i.e.* a closed string theory. The main question we want to address is: how can we extend this conformal field theory to a theory that is also defined on world-sheets with boundary. More precisely we want to ask which boundary conditions can be imposed at the various boundaries. From a string theory point of view, this is the question of which open strings can be consistently added to a given closed string theory.

### 2.1 Generalities

In some sense, this problem is rather similar to a familiar construction in (closed) conformal field theory. Suppose we are given the theory defined on the sphere. We can then ask whether this theory determines already (uniquely) the theory on arbitrary Riemann surfaces. The answer is well known [24, 25, 26]: the theory on the sphere determines uniquely the theory on an arbitrary closed Riemann surface (if it exists), but it does not guarantee that it is consistent. Indeed, there is one additional consistency condition that arises at genus 1 (and that does not follow from the consistency of the theory on the sphere), namely that the correlation functions on the torus transform under the action of the modular group  $SL(2, \mathbb{Z})$ .<sup>2</sup> If this consistency condition is satisfied, the theory is consistent on all Riemann surfaces [26].

The analogous result for the construction of the theory on surfaces with boundaries is not known. For a given theory defined on the sphere, the complete list of ‘sewing relations’ that have to be satisfied by each boundary condition is known [27, 28]. However it is not clear for which classes of theories solutions to these sewing relations can be found, and if so, how many. Based on the examples that have been understood [29, 30, 31, 32, 33, 34] it appears that modular invariance may again be sufficient to guarantee that a ‘complete’ set of boundary conditions can be constructed. In fact, there are striking similarities between the classification of modular invariant partition functions and that of the so-called NIM-reps (non-negative integer matrix representations of the fusion algebra) that appear naturally in the construction of the boundary states [31, 35]. On the other hand, it seems that there are more NIM-reps than (consistent) conformal field theories that can be defined on the torus, and at least some of the additional NIM-reps seem to be naturally related to consistent conformal field theories that are only defined on the sphere (but not on the torus).

The basic reason why the theory on the sphere determines already the theory on all Riemann surfaces can be schematically understood as follows. Since we are dealing with

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<sup>2</sup>For example, the theory of a single NS fermion is consistent on the sphere but does not satisfy the modular consistency condition.

a local conformal field theory, the operator product expansion of any two operators is the same, irrespective of the surrounding surface. The operator product expansion (and thus the ‘local structure’ of the theory) is therefore already determined by the theory on the sphere.

For the problem we are actually interested in, namely the extension of the theory on the sphere, say, to surfaces with boundary, a similar consideration applies. As we have just explained, given the theory on the sphere we can deduce the operator product expansion of the fields  $\phi_a$ , which we can write schematically as

$$\phi_a \phi_b \sim \sum_c C_{ab}^c \phi_c. \quad (2.1)$$

Here  $C_{ab}^c$  are the structure constants of the theory on the sphere, and we have suppressed the dependence of the fields on the coordinates on the sphere. We can think of the operator product expansion as defining an ‘algebra of fields’.<sup>3</sup> The boundary conditions we are interested in have to respect this algebra, and they must therefore define an ‘algebra homomorphism’

$$(\text{‘algebra of fields’}) \longrightarrow \mathbb{C}. \quad (2.2)$$

Every element of the space of states of the theory on the sphere  $\mathcal{H}$  defines a map of the form (2.2), and in fact every such map arises from a suitable (infinite) linear combination of such states. Thus we can describe each boundary condition by a ‘coherent’ *boundary state* in  $\mathcal{H}^4$ , and for the boundary condition labelled by  $\alpha$  we denote the corresponding boundary state by  $\|\alpha\rangle\rangle$ . Given this boundary state, the amplitudes of the fields in the presence of the boundary with boundary condition  $\alpha$  are then simply given by the (closed string) expression

$$\langle \phi_1 \phi_2 \phi_3 \rangle_\alpha = \langle \phi_1 \phi_2 \phi_3 \|\alpha\rangle\rangle. \quad (2.3)$$

## 2.2 Gluing conditions

Not every linear map of the form (2.2) actually defines a boundary state. (Indeed, it follows from the above discussion, that there exists for example a coherent state for each higher genus Riemann surface.) The coherent states that describe boundary conditions are characterised by the property that the left- and right-moving fields corresponding to unbroken symmetries are related to one another at the boundary. If we take the boundary to be along the real axis, the relevant condition is that

$$S(z) = \rho(\bar{S}(\bar{z})) \quad \text{for } z \in \mathbb{R}, \quad (2.4)$$

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<sup>3</sup>Because of the dependence on the coordinates, this is not really an algebra, but rather (a slight generalisation of) what is usually called a *vertex operator algebra*.

<sup>4</sup>As we shall see momentarily, the boundary states are necessarily coherent states, *i.e.* they do not lie in the Fock space of finite energy states.

where  $S$  and  $\bar{S}$  are generators of the symmetry that is preserved by the boundary, and  $\rho$  denotes an automorphism of the algebra of fields that leaves the stress-energy tensor invariant. The fields  $S(z)$  and  $\bar{S}(\bar{z})$  have an expansion in terms of modes as

$$S(z) = \sum_{n \in \mathbb{Z}} S_n z^{-n-h}, \quad \bar{S}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{S}_n \bar{z}^{-n-h}, \quad (2.5)$$

where  $h$  is the conformal weight of  $S$  (and  $\bar{S}$ ). In the description of the boundary condition in terms of a boundary state, the boundary is taken to be the (unit) circle around the origin. In order to express the above condition in terms of a condition involving the boundary state, we apply the conformal transformation that maps the upper half plane to the disc

$$\zeta(z) = \frac{z+i}{z-i}, \quad (2.6)$$

together with its complex conjugate  $\bar{\zeta} = \frac{\bar{z}-i}{\bar{z}+i}$ . Since

$$\zeta'(z) = \frac{1}{2}(\zeta-1)^2, \quad \bar{\zeta}'(\bar{z}) = -\frac{1}{2}(\bar{\zeta}-1)^2, \quad (2.7)$$

it follows that

$$\left(\frac{1}{2}(\zeta-1)^2\right)^h S(\zeta) = \left(-\frac{1}{2}(\bar{\zeta}-1)^2\right)^h \rho(\bar{S}(\bar{\zeta})) \quad \text{for } |\zeta| = 1, \quad (2.8)$$

where we have used that a (primary) conformal field transforms as

$$S(z) \mapsto \zeta'(z)^h S(\zeta(z)), \quad (2.9)$$

and similarly for  $\bar{S}$ . For  $|\zeta| = 1$ ,  $\bar{\zeta} = \zeta^{-1}$ , and thus

$$(\zeta-1)^{2h} = \zeta^{2h} (\bar{\zeta}-1)^{2h}, \quad (2.10)$$

where we have assumed that  $h \in \mathbb{Z}$ . [Alternatively, the factor  $(-1)^h$  would be replaced by  $(-1)^{h+2h}$  below.] Using (2.5) we thus find that a boundary state  $|\alpha\rangle$  that preserves the symmetry described by  $S$  has to satisfy

$$\left( \sum_{n \in \mathbb{Z}} S_n \zeta^{h-n} - (-1)^h \sum_{n \in \mathbb{Z}} \rho(\bar{S}_n) \zeta^{n+h} \right) |\alpha\rangle = 0 \quad \text{for } |\zeta| = 1. \quad (2.11)$$

Since this has to hold for all  $\zeta$  with  $|\zeta| = 1$ , (2.11) implies the so-called ‘gluing condition’

$$(S_n - (-1)^h \rho(\bar{S}_{-n})) |\alpha\rangle = 0 \quad \text{for all } n \in \mathbb{Z}. \quad (2.12)$$

The gluing condition implies in particular that  $\|\alpha\rangle\rangle$  must be a coherent state.

There is only one symmetry that every boundary condition *has* to preserve. This is the ‘conformal’ symmetry that guarantees that the resulting field theory is again conformal. In terms of (2.12) it corresponds to the gluing condition

$$(L_n - \bar{L}_{-n}) \|\alpha\rangle\rangle = 0 \quad \text{for all } n \in \mathbb{Z}, \quad (2.13)$$

where  $L_n$  and  $\bar{L}_n$  are the modes of the left- and right-moving stress energy tensor of conformal weight  $h_L = h_{\bar{L}} = 2$ . Recently some progress has been made in understanding the ‘conformal’ branes (*i.e.* the branes that only preserve the conformal symmetry) in a few simple examples [36, 37, 38, 39], but in general very little is known about this problem. In most cases, however, the D-branes that preserve additional symmetries account for all the K-theory charges of the theory, and it is often therefore sufficient to concentrate on those.

### 2.3 The rational case

The more symmetries we require the boundary condition to preserve, the fewer boundary conditions exist, and the more tractable the problem becomes. The situation is particularly simple if the closed theory is a ‘rational’ theory with respect to the preserved symmetry algebra: let us assume we are interested in boundary conditions that respect the symmetry algebra  $\mathcal{A}$  (where we take, for simplicity,  $\rho = \text{id}$ ). In order to determine the relevant boundary conditions we decompose the space of states of the closed string theory  $\mathcal{H}$  in terms of representations of  $\mathcal{A} \otimes \bar{\mathcal{A}}$  as

$$\mathcal{H} = \bigoplus_{i,j} N_{ij} \mathcal{H}_i \otimes \bar{\mathcal{H}}_j, \quad (2.14)$$

where the sum runs over the set of irreducible representations of  $\mathcal{A}$  and  $\bar{\mathcal{A}} \cong \mathcal{A}$ , and  $N_{ij}$  describes the multiplicity with which the irreducible representation  $\mathcal{H}_i \otimes \bar{\mathcal{H}}_j$  of  $\mathcal{A} \otimes \bar{\mathcal{A}}$  appears in  $\mathcal{H}$ . The theory is called ‘rational with respect to  $\mathcal{A}$ ’ if  $\mathcal{A}$  only possesses finitely many irreducible representations. In this case, the sum in (2.14) is finite. The vacuum representation is denoted by  $\mathcal{H}_0$ ; the uniqueness of the vacuum implies that  $N_{00} = 1$ . In the following we assume some basic familiarity with conformal field theory (see for example [40, 41] for some suitable reviews).

Since the modes that appear in the gluing condition (2.12) map each  $\mathcal{H}_i \otimes \bar{\mathcal{H}}_j$  into itself, we can solve the gluing constraint separately for each summand in (2.14). We can find a non-trivial solution provided that  $\mathcal{H}_i$  is the conjugate representation of  $\bar{\mathcal{H}}_j$ . If this is the case, there is (up to normalisation) only one coherent state that satisfies

(2.12);<sup>5</sup> this state is called the *Ishibashi state* [42] and it is denoted by

$$|i\rangle\rangle \in \mathcal{H}_i \otimes \bar{\mathcal{H}}_i, \quad \left( S_n - (-1)^{h_S} \rho(\bar{S}_{-n}) \right) |i\rangle\rangle = 0 \quad \text{for all } n \in \mathbb{Z} \text{ and } S \in \mathcal{A}. \quad (2.15)$$

If the theory is rational then there are in particular only finitely many Ishibashi states.

Since every boundary state satisfies the gluing condition (2.12) it must be a linear combination of the Ishibashi states. We can therefore write every boundary state as

$$|\alpha\rangle\rangle = \sum_i \frac{\psi_{\alpha i}}{\sqrt{S_{0i}}} |i\rangle\rangle, \quad (2.16)$$

where  $\psi_{\alpha i}$  are some constants that characterise the boundary condition, and  $S_{0i}$  denotes the modular  $S$ -matrix (see (2.24) below). The constants  $\psi_{\alpha i}$  are constrained by two classes of conditions:

- The Cardy condition [43].
- The so-called ‘sewing relations’ that were first derived in [28, 27].

## 2.4 The Cardy condition

The Cardy condition comes about as follows. Let us consider the (open string) partition function

$$Z_{\alpha\beta}(\tilde{q}) = \text{Tr}_{\mathcal{H}_{\alpha\beta}} e^{-2\pi T/L H_{\alpha\beta}} = \sum_i \mathcal{N}_{i\beta}^\alpha \chi_i(\tilde{q}) \quad (2.17)$$

of the open string with boundary conditions  $\alpha$  and  $\beta$  at the two ends. Here  $\mathcal{H}_{\alpha\beta}$  is the corresponding space of open string states, and  $H_{\alpha\beta}$  the relevant Hamilton operator — the factor of  $1/L$  is sometimes part of the definition of  $H_{\alpha\beta}$ , but for the following it is useful to make the dependence on  $L$  explicit. In writing the second equation in (2.17) we have used that the boundary conditions preserve  $\mathcal{A}$ , and therefore that we can decompose  $\mathcal{H}_{\alpha\beta}$  with respect to  $\mathcal{A}$  as

$$\mathcal{H}_{\alpha\beta} = \bigoplus_i \mathcal{N}_{i\beta}^\alpha \mathcal{H}_i, \quad (2.18)$$

where each  $\mathcal{H}_i$  is an irreducible representation of  $\mathcal{A}$ . The numbers  $\mathcal{N}_{i\beta}^\alpha$  describe the multiplicity with which  $\mathcal{H}_i$  appears in  $\mathcal{H}_{\alpha\beta}$ , and they are therefore non-negative integers. (In fact, as we shall see below, the numbers  $\mathcal{N}_{i\beta}^\alpha$  are precisely the entries of the NIM-reps we mentioned before.) We have furthermore used the usual short hand notation for the character of a representation,

$$\chi_i(\tilde{q}) = \text{Tr}_{\mathcal{H}_i} \left( e^{-2\pi T/L H_{\alpha\beta}} \right), \quad \tilde{q} = e^{-2\pi T/L}. \quad (2.19)$$

Figure 1: World-sheet duality

In terms of the boundary states we introduced before (*i.e.* from the closed string point of view) this amplitude is simply the overlap

$$\begin{aligned} Z_{\alpha\beta}(\tilde{q}) &= \langle\langle \alpha \| e^{-2\pi L/T H_{cl}} \| \beta \rangle\rangle \\ &= \sum_i \frac{\psi_{\alpha i}^* \psi_{\beta i}}{S_{0i}} \chi_i(q). \end{aligned} \quad (2.20)$$

Here  $H_{cl}$  is the closed string Hamiltonian (again, the factor  $1/T$  is sometimes part of the definition of  $H_{cl}$ ), and we have used (2.16) to write the boundary states in terms of the Ishibashi states. We have furthermore used that

$$\langle\langle i | e^{-2\pi L/T H_{cl}} | j \rangle\rangle = \delta_{ij} \chi_i(q), \quad (2.21)$$

where  $\chi_i(q)$  is again the character of the representation  $\mathcal{H}_i$  that is now evaluated at  $q$  with  $q = e^{-2\pi L/T}$  rather than  $\tilde{q}$ . If we write

$$q = e^{2\pi i\tau}, \quad \tau = iL/T, \quad (2.22)$$

then  $\tilde{q}$  is simply given as

$$\tilde{q} = e^{-\frac{2\pi i}{\tau}}. \quad (2.23)$$

Thus  $q$  and  $\tilde{q}$  are related by the standard modular  $S$ -transformation that maps  $\tau \mapsto -1/\tau$ . At least for rational conformal field theories (and in fact under certain slightly

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<sup>5</sup>We are assuming here, for ease of notation, that the multiplicities  $N_{ij}$  are all either zero or one; the modifications for the general case are obvious.

weaker conditions) the characters of the irreducible representations transform into one another as

$$\chi_i(q) = \sum_j S_{ji} \chi_j(\tilde{q}), \quad (2.24)$$

where  $S_{ij}$  is the symmetric and unitary matrix representing the  $S$ -transformation of the modular group  $\text{SL}(2, \mathbf{Z})$ . Inserting (2.24) into (2.20) we therefore find that

$$Z_{\alpha\beta}(\tilde{q}) = \sum_{i,j} \frac{\psi_{\alpha i}^* \psi_{\beta i} S_{ji}}{S_{0i}} \chi_j(\tilde{q}). \quad (2.25)$$

Comparing with (2.17), and assuming that the characters of the irreducible representations are linearly independent, it therefore follows that

$$\mathcal{N}_{j\beta}^\alpha = \sum_i \frac{\psi_{\alpha i}^* \psi_{\beta i} S_{ji}}{S_{0i}}. \quad (2.26)$$

This is a very restrictive condition that is often (in particular, if the theory is rational and there are only finitely many irreducible representations) fairly accessible. It requires that every set of consistent boundary states gives rise to a family  $\mathcal{N}_{j\beta}^\alpha$  of Non-negative Integer Matrices (NIM), one for each representation  $j$ .

The set of solutions to Cardy's condition form (the positive cone of) a lattice: suppose that the set

$$M = \{\|\alpha_1\rangle\rangle, \dots, \|\alpha_n\rangle\rangle\} \quad (2.27)$$

satisfies Cardy's condition, *i.e.* the overlap between any two elements of  $M$  leads to non-negative integer numbers  $\mathcal{N}_{i,\alpha_k}^{\alpha_j}$ , then so does the set

$$M' = \left\{ \|\alpha_1\rangle\rangle, \dots, \|\alpha_n\rangle\rangle, \sum_{l=1}^n m_l \|\alpha_l\rangle\rangle \right\}, \quad (2.28)$$

provided that  $m_l \in \mathbb{N}_0$  for  $l = 1, \dots, n$ . This is simply a consequence of the fact that sums of products of non-negative integers are non-negative integers. What we therefore want to find are the *fundamental* boundary conditions that generate all other boundary conditions upon taking positive integer linear combinations as above.

These fundamental boundary conditions are believed to be characterised by the condition that the  $\psi_{\alpha i}$  actually form a *unitary* matrix, *i.e.* that

$$\sum_\alpha \psi_{\alpha i}^* \psi_{\alpha j} = \delta_{ij}. \quad (2.29)$$

(In particular, there are then as many boundary states as Ishibashi states.) If this is the case, the NIM-numbers (2.26) actually form a *representation of the fusion algebra*

(or NIM-rep for short). This is to say,

$$\begin{aligned}
\sum_{\beta} \mathcal{N}_{i\beta}^{\alpha} \mathcal{N}_{j\gamma}^{\beta} &= \sum_{\beta} \sum_{lm} \frac{\psi_{\alpha l}^* S_{il} \psi_{\beta l}}{S_{0l}} \frac{\psi_{\beta m}^* S_{jm} \psi_{\gamma m}}{S_{0m}} \\
&= \sum_l \frac{\psi_{\alpha l}^* S_{il} S_{jl} \psi_{\gamma l}}{S_{0l} S_{0l}} \\
&= \sum_{lkm} \frac{\psi_{\alpha l}^* S_{kl} \psi_{\gamma l}}{S_{0l}} \frac{S_{km}^* S_{im} S_{jm}}{S_{0m}} \\
&= \sum_k N_{ij}^k \mathcal{N}_{k\gamma}^{\alpha}, \tag{2.30}
\end{aligned}$$

where  $N_{ij}^k$  are the *fusion rules* of the theory, that are described by the Verlinde formula [44]

$$N_{ij}^k = \sum_m \frac{S_{km}^* S_{im} S_{jm}}{S_{0m}}. \tag{2.31}$$

In the penultimate line of (2.30) we have used that the  $S$ -matrix is unitary: in particular, performing the sum over  $k$  leads to  $\delta_{lm}$ .

In general it is not known how to find the coefficients  $\psi_{\alpha i}$ , or the corresponding NIM-rep. However, there is one class of theories, where the answer is known in general. These are the diagonal (modular invariant) theories whose spectrum is characterised by  $N_{ij} = \delta_{ij}$ . In this case, there are as many Ishibashi states as there are irreducible representations of the chiral algebra, and therefore also as many boundary states. The boundary states can then be labelled by the irreducible representations, and they are explicitly given as [43]

$$|\alpha_j\rangle\rangle = \sum_i \frac{S_{ji}}{\sqrt{S_{0i}}} |i\rangle\rangle, \tag{2.32}$$

*i.e.* by  $\psi_{\alpha_j i} = S_{ji}$ . For these boundary states, the NIM-rep is just the fusion algebra itself since (2.26) then reduces to (2.31).

## 2.5 A sewing relation

The boundary states also have to satisfy a number of sewing relations [28, 27]. In the following we want to discuss, as an example, one of these sewing relations; we shall not discuss any of the other sewing relations in these lectures.

Suppose that we have found a solution to Cardy's condition. If the boundary states are fundamental (*i.e.* if the  $\psi$ -matrix is unitary) then the numbers  $\mathcal{N}_{i\beta}^{\alpha}$  actually form a NIM-rep, and therefore  $\mathcal{N}_{0\alpha}^{\alpha} = 1$ . This means that the self-overlap of each boundary state

$|\alpha\rangle\rangle$  contains the vacuum representation in the open string channel with multiplicity one, *i.e.*

$$\langle\langle\alpha| e^{-2\pi LH_{cl}} |\alpha\rangle\rangle = \chi_0(\tilde{q}) + \dots . \quad (2.33)$$

Provided this is the case, one of the sewing relations simplifies considerably, and actually gives rise to a powerful constraint (see [36] for a more detailed derivation). This constraint arises from considering a two-point function of primary fields in the presence of such a boundary condition,

$$F_{ab}(z, \bar{z}, w, \bar{w}) = \langle \varphi_a(z, \bar{z}) \varphi_b(w, \bar{w}) \rangle. \quad (2.34)$$

The gluing conditions for the energy-momentum tensor imply that (2.34) can be described in terms of four-point chiral blocks where we insert chiral vertex operators of weight  $h_a, \bar{h}_a, h_b$  and  $\bar{h}_b$  at  $z, \bar{z}, w$  and  $\bar{w}$ , respectively. This four-point function can then be factorised in two different ways, leading to two different representations of the correlation function, as shown below. In the first picture one considers the limit in which the two fields approach the boundary separately; in the second picture on the other hand, the two fields come close together away from the boundary, and we can thus use the operator product expansion (2.1) in order to express the product of these two fields in terms of a sum of single fields:

$$\sim \frac{\psi_{\alpha a}}{\sqrt{S_{0a}}} \frac{\psi_{\alpha b}}{\sqrt{S_{0b}}} |z - \bar{z}|^{2h_b - 2h_a} |z - \bar{w}|^{-4h_b} f^1 \left[ \begin{matrix} a b \\ a b \end{matrix} \right] (\eta). \quad (2.35)$$

$$\sim \sum_c C_{ab}^c \frac{\psi_{\alpha c}}{\sqrt{S_{0c}}} |z - \bar{z}|^{2h_b - 2h_a} |z - \bar{w}|^{-4h_b} f^c \left[ \begin{matrix} b b \\ a a \end{matrix} \right] (1 - \eta). \quad (2.36)$$

In writing down these equations we have specialised to the case where  $\varphi_a$  and  $\varphi_b$  are self-conjugate fields for which  $h_a = \bar{h}_a$  and  $h_b = \bar{h}_b$ . The  $f^1$  and  $f^c$  denote the different

chiral four-point blocks, and  $\eta$  is the cross-ratio  $\eta = |(z-w)/(z-\bar{w})|^2$  which is real with  $0 \leq \eta \leq 1$ . In both equations we have only considered the leading behaviour as  $\eta \rightarrow 1$ , *i.e.* we have only taken into consideration the contribution of the vacuum state in the open string channel.<sup>6</sup> (This is where we have used the assumption that the boundary condition in question is fundamental to deduce that there is only one such state.) The two sets of chiral blocks are related by the so-called fusing matrices

$$f^c \begin{bmatrix} bb \\ aa \end{bmatrix} (1 - \eta) = F_{c1} \begin{bmatrix} bb \\ aa \end{bmatrix} f^1 \begin{bmatrix} ab \\ ab \end{bmatrix} (\eta). \quad (2.37)$$

Substituting (2.37) in (2.36) and comparing with (2.35), we then obtain the sewing relation

$$\frac{\psi_{\alpha a}}{\sqrt{S_{0a}}} \frac{\psi_{\alpha b}}{\sqrt{S_{0b}}} = \sum_c C_{ab}{}^c F_{c1} \begin{bmatrix} bb \\ aa \end{bmatrix} \frac{\psi_{\alpha c}}{\sqrt{S_{0c}}}. \quad (2.38)$$

This condition is known as the ‘factorisation constraint’ [28, 27], the ‘cluster condition’ [45] or the ‘classifying algebra’ [46]. In many cases it is, however, rather difficult to check since the structure constants  $C$  and  $F$  on the right hand side are often not explicitly known. However, there are a few examples where one can actually determine the structure of (2.38) explicitly, and use this to classify all possible fundamental D-branes of a theory (see for example [36, 37]).

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<sup>6</sup>We are assuming here that the state of lowest conformal weight in the open string spectrum is the vacuum state. The generalisation to non-unitary theories where this is not the case is straightforward.

### 3 D-branes in string theory

We now want to describe how the techniques that were described in the previous section can be used to construct D-branes in string theory. From a string theory point of view the simplest example is the uncompactified 26-dimensional free bosonic string. This is not described by a rational conformal field theory, but it can nevertheless be treated by these methods.

#### 3.1 The bosonic string

From the point of view of the world-sheet theory, the bosonic string consists of 26 free bosonic fields  $X^\mu(\sigma, \tau)$ ,  $\mu = 0, \dots, 25$  that describe the embedding of the string world-sheet in the target space. Here  $\sigma$  and  $\tau$  are the space and time-coordinate on the world-sheet. We shall consider here the situation where the target space is flat, uncompactified 26-dimensional Minkowski space with metric  $\eta^{\mu\nu}$ . The equations of motion imply that we can expand the string fields  $X^\mu(\sigma, \tau)$  as

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma), \quad (3.1)$$

where in terms of modes,

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau + \sigma)} \quad (3.2)$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\tau - \sigma)}. \quad (3.3)$$

The canonical equal-time commutation relations for the fields  $X^\mu(\tau, \sigma)$  are

$$[X^\mu(\tau, \sigma), \partial_\tau X^\nu(\tau, \sigma')] = \pi i \eta^{\mu\nu} \delta(\sigma - \sigma'), \quad (3.4)$$

and this implies that the modes satisfy the commutation relations

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= m \eta^{\mu\nu} \delta_{m, -n} \\ [\alpha_m^\mu, \tilde{\alpha}_n^\nu] &= 0 \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= m \eta^{\mu\nu} \delta_{m, -n}. \end{aligned} \quad (3.5)$$

In addition, the zero mode  $x^\mu$  commutes with all  $\alpha_n^\nu$  and  $\tilde{\alpha}_n^\nu$  (for  $n \neq 0$ ), and satisfies

$$[x^\mu, p^\nu] = \frac{i}{2} \eta^{\mu\nu}. \quad (3.6)$$

In order to relate this description to the conformal field theory discussion of the previous section we Wick rotate the world-sheet theory, replacing  $\tau$  by  $-i\tau$ . Furthermore, we write

$$z = e^{\tau+i\sigma}, \quad \bar{z} = e^{\tau-i\sigma}, \quad (3.7)$$

and then  $X_L^\mu$  and  $X_R^\mu$  become functions of  $z$  and  $\bar{z}$ , respectively,

$$X_L^\mu(z) = \frac{1}{2}x^\mu - \frac{i}{2}p^\mu \log z + \frac{i}{2} \sum_{n \neq 0} \frac{z^{-n}}{n} \alpha_n^\mu \quad (3.8)$$

$$X_R^\mu(\bar{z}) = \frac{1}{2}x^\mu - \frac{i}{2}p^\mu \log \bar{z} + \frac{i}{2} \sum_{n \neq 0} \frac{\bar{z}^{-n}}{n} \tilde{\alpha}_n^\mu. \quad (3.9)$$

While the bosonic fields themselves are not conformal primary fields, their derivatives are

$$\partial_z X_L^\mu(z) = -\frac{i}{2} \sum_{n \in \mathbb{Z}} \alpha_n^\mu z^{-n-1}, \quad (3.10)$$

$$\partial_{\bar{z}} X_R^\mu(\bar{z}) = -\frac{i}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_n^\mu \bar{z}^{-n-1}, \quad (3.11)$$

where we have defined  $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu$ . These fields are then conformal primary fields of conformal weight  $h = 1$ ; their modes satisfy a  $u(1)$  current algebra (3.5). The conformal fields  $L$  and  $\tilde{L}$  are quadratic in the bosonic fields  $X_L^\mu$  and  $X_R^\mu$ , respectively,

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \eta_{\mu\nu} : \alpha_m^\mu \alpha_{n-m}^\nu : \quad (3.12)$$

$$\tilde{L}_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \eta_{\mu\nu} : \tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m}^\nu :, \quad (3.13)$$

where the colons denote normal ordering, *i.e.* creation operators  $\alpha_n^\mu$  with  $n > 0$  are written to the left of annihilation operators  $\alpha_n^\mu$  with  $n < 0$ , and similarly for  $\tilde{\alpha}_n^\mu$ .

The possible representations of this conformal field theory are labelled by the momentum  $\mathbf{k}$  of its ground state. More precisely, the sector labelled by  $\mathbf{k}$  is generated by the action of the creation operators  $\alpha_{-n}^\mu$  and  $\tilde{\alpha}_{-n}^\mu$  with  $n > 0$  from the highest weight state  $|\mathbf{k}\rangle$  that is characterised by the properties

$$\begin{aligned} \alpha_n^\mu |\mathbf{k}\rangle &= k^\mu \delta_{n,0} & \text{for } n \geq 0 \\ \tilde{\alpha}_n^\mu |\mathbf{k}\rangle &= k^\mu \delta_{n,0} & \text{for } n \geq 0. \end{aligned} \quad (3.14)$$

Since  $\mathbf{k}$  can be any vector, there are infinitely many highest weight representations, and the theory is therefore not rational. Nevertheless, the analysis of the previous subsection can still be performed with only minor modifications.

If we analyse the boundary states that preserve the full current symmetry<sup>7</sup>, we are looking for boundary states that satisfy

$$\left(\alpha_l^\mu + \rho(\tilde{\alpha}_{-l}^\mu)\right) \|\alpha\rangle\rangle = 0 \quad \text{for all } l \in \mathbb{Z}. \quad (3.15)$$

For each  $\mu$  there are two possible choices for  $\rho$ : either  $\rho$  is the identity, or  $\rho(\tilde{\alpha}_{-l}^\mu) = -\tilde{\alpha}_{-l}^\mu$ . The former case corresponds thus to the gluing condition

$$\left(\alpha_l^\mu + \tilde{\alpha}_{-l}^\mu\right) \|N\rangle\rangle = 0 \quad \text{for all } l \in \mathbb{Z}. \quad (3.16)$$

This is usually called the **Neumann** boundary condition. Indeed, if we write this in terms of the field  $X^\mu(\tau, \sigma)$ , then this condition means that

$$\partial_\tau X^\mu(\tau, \sigma)|_{\tau=0} \|N\rangle\rangle = \frac{1}{2} \sum_{l \in \mathbb{Z}} e^{-il\sigma} \left(\alpha_l^\mu + \tilde{\alpha}_{-l}^\mu\right) \|N\rangle\rangle = 0. \quad (3.17)$$

[Note that from the closed string point of view, the boundary is at  $\tau = 0$ ; the normal derivative is therefore the  $\tau$ -derivative.]

On the other hand, the other choice for  $\rho$  leads to the **Dirichlet** boundary condition

$$\left(\alpha_l^\nu - \tilde{\alpha}_{-l}^\nu\right) \|D\rangle\rangle = 0 \quad \text{for all } l \in \mathbb{Z}. \quad (3.18)$$

Together with the zero-mode condition

$$x^\nu \|D\rangle\rangle = a^\nu \|D\rangle\rangle, \quad (3.19)$$

where  $a^\mu$  is a constant, this corresponds then to the boundary condition

$$X^\nu(\tau, \sigma)|_{\tau=0} \|D\rangle\rangle = a^\nu \|D\rangle\rangle. \quad (3.20)$$

The general case is thus described by choosing a Neumann or Dirichlet boundary condition for each direction. Since the theory has an  $SO(25, 1)$  symmetry, we may without loss of generality assume that the first  $p + 1$  directions are Neumann directions, while the remaining  $25 - p$  directions are Dirichlet. The resulting boundary condition is then called a **Dp-brane**. It describes a  $p + 1$ -dimensional hypersurface that is embedded in the ambient space by setting  $x^\nu = a^\nu$  for each of the  $25 - p$  transverse directions.

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<sup>7</sup>In the following we shall only consider the case where each of the twenty-six different  $u(1)$  symmetries is separately preserved. The full current symmetry would also be preserved if the twenty-six left- and right-moving currents were related by a rotation in  $SO(25, 1)$ . A general brane of this type would then typically carry non-trivial world-volume fluxes.

In the following we shall always work in light-cone gauge. To this end we introduce the light-cone fields

$$X^\pm(\tau, \sigma) = \frac{1}{\sqrt{2}} \left( X^0(\tau, \sigma) \pm X^{25}(\tau, \sigma) \right), \quad (3.21)$$

and likewise for the modes. We can fix the reparametrisation invariance of the world-sheet theory by choosing  $X^+$  to be proportional to the world-sheet time parameter  $\tau$ , *i.e.*  $X^+ = 2\pi\alpha'p^+\tau$ . Since the boundary is inserted at  $\tau = 0$ , this means that one automatically chooses a Dirichlet boundary condition for  $x^+$ . Via the constraint equations that determine  $X^-$ , it furthermore follows that this also imposes a Dirichlet boundary condition for  $X^-$ . Thus the D-branes we shall construct in the following are really D-instantons since they satisfy a Dirichlet boundary condition in time. However, our results can be related to the more usual time-like Dirichlet branes by performing a double Wick rotation [47].

### 3.1.1 The explicit boundary state and the Cardy condition

Since the field theory is actually free, it is not difficult to write down the Ishibashi states and the boundary state explicitly. To fix our notation, let us assume that the boundary state should satisfy

$$\begin{aligned} \left( \alpha_l^\mu + \tilde{\alpha}_{-l}^\mu \right) \|Bp, \mathbf{a}\rangle &= 0 & \mu = 1, \dots, p+1 \\ \left( \alpha_l^\nu - \tilde{\alpha}_{-l}^\nu \right) \|Bp, \mathbf{a}\rangle &= 0 & \nu = p+2, \dots, 24 \\ x^\nu \|Bp, \mathbf{a}\rangle &= a^\nu \|Bp, \mathbf{a}\rangle & \nu = p+2, \dots, 24 \end{aligned} \quad (3.22)$$

The first condition with  $l = 0$  implies that only highest weight representations for which  $k^\mu = 0$  for  $\mu = 1, \dots, p+1$  can support an Ishibashi state. If this is the case, the relevant Ishibashi state is simply of the form

$$|Bp, \mathbf{k}\rangle = \exp \left\{ \sum_{n>0} \left[ -\frac{1}{n} \sum_{\mu=1}^{p+1} \alpha_{-n}^\mu \tilde{\alpha}_{-n}^\mu + \frac{1}{n} \sum_{\nu=p+2}^{24} \alpha_{-n}^\nu \tilde{\alpha}_{-n}^\nu \right] \right\} |\mathbf{k}\rangle. \quad (3.23)$$

In fact, it is easy to see, given the commutation relations (3.5), that (3.23) satisfies the first two equations of (3.22). In order to satisfy the last condition, we have to consider a suitable linear superposition of Ishibashi states — this is the analogue of (2.32). In fact, given (3.6), the full boundary state is simply the Fourier transform,

$$\|Bp, \mathbf{a}\rangle = \mathcal{N} \int \prod_{\nu=p+2, \dots, 24} dk^\nu e^{ik^\nu a^\nu} |Bp, \mathbf{k}\rangle. \quad (3.24)$$

The normalisation constant  $\mathcal{N}$  is determined by the analogue of the Cardy condition. To this end, we determine the closed string overlap of two such boundary states,

$$\mathcal{A} = \langle\langle Bp, \mathbf{a}_1 \| e^{-tH_c} \| Bp, \mathbf{a}_2 \rangle\rangle, \quad (3.25)$$

and  $H_c$  is the closed string Hamiltonian in light cone gauge, *i.e.*

$$H_c = \frac{1}{2}\pi\mathbf{k}^2 + \pi \sum_{\mu=1, \dots, 24} \left[ \sum_{n=1}^{\infty} (\alpha_{-n}^{\mu} \alpha_n^{\mu} + \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_n^{\mu}) \right] - 2\pi. \quad (3.26)$$

(Here the last term in  $H_c$  is the usual normal ordering constant in bosonic string theory.) Given the explicit form of the boundary state, it is easy to work out this overlap, and one finds

$$\mathcal{A} = \mathcal{N}^2 2^{\frac{23-p}{2}} t^{-\frac{23-p}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2}{2\pi t}} \frac{1}{f_1(q)^{24}}, \quad (3.27)$$

where  $q = e^{-2\pi t}$ , and the  $f_i$  functions are defined as in [48]

$$\begin{aligned} f_1(q) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \\ f_2(q) &= \sqrt{2} q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n), \\ f_3(q) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 + q^{n-1/2}), \\ f_4(q) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{n-1/2}). \end{aligned} \quad (3.28)$$

Under the modular transformation,  $\tilde{t} = 1/t$ , the  $f_i$  functions transform as

$$\begin{aligned} f_1(q) &= t^{-1/2} f_1(\tilde{q}), & f_2(q) &= f_4(\tilde{q}), \\ f_3(q) &= f_3(\tilde{q}), & f_4(q) &= f_2(\tilde{q}), \end{aligned} \quad (3.29)$$

where  $\tilde{q} = e^{-2\pi\tilde{t}} = e^{-2\pi/t}$ . The amplitude  $\mathcal{A}$  can thus be rewritten as

$$\mathcal{A} = \mathcal{N}^2 2^{\frac{23-p}{2}} \tilde{t}^{-\frac{p+1}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2}{2\pi} \tilde{t}} \frac{1}{f_1(\tilde{q})^{24}}. \quad (3.30)$$

This should now be interpreted as the open-string trace

$$\mathcal{Z} = \text{Tr}_{\mathcal{H}_{D_p, D_p}} \left( e^{-2\tilde{t}H_o} \right), \quad (3.31)$$

where  $H_o$  is the open string Hamiltonian in light-cone gauge,

$$H_o = \pi \vec{p}^2 + \frac{1}{4\pi} \vec{w}^2 + \pi \sum_{\mu=1, \dots, 24} \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_n^{\mu} - \pi. \quad (3.32)$$

Here  $\vec{p}$  denotes the open string momentum along the directions for which the string has Neumann (N) boundary conditions,  $\vec{w}$  is the difference between the two end-points of the open string (along the Dirichlet directions), and  $\alpha_n^{\mu}$  are the open string modes which satisfy the same commutation relations as (3.5) above. The trace includes an integral over the open string momenta for the Neumann directions. In our case,  $\vec{w} = (\mathbf{a}_1 - \mathbf{a}_2)$ , and the integral over the Neumann directions gives  $(2\tilde{t})^{-\frac{p+1}{2}}$ , thus leading to

$$\mathcal{Z} = (2\tilde{t})^{-\frac{p+1}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2 \tilde{t}}{2\pi}} \frac{1}{f_1(\tilde{q})^{24}}. \quad (3.33)$$

This agrees with  $\mathcal{A}$  above provided that

$$\mathcal{N}^2 2^{\frac{23-p}{2}} = 2^{-\frac{p+1}{2}}, \quad \mathcal{N} = 2^{-6}. \quad (3.34)$$

### 3.1.2 The compactified case

The analysis is actually slightly cleaner if one considers the situation where the target space is compactified on some torus. In the simplest case, this torus is just an orthogonal torus for which the different directions decouple. We may then, without loss of generality, consider each direction by itself, and thus study the theory whose target space (in light-cone gauge) is just a circle of radius  $R$ .

The main effect of the circle compactification is to restrict the possible momenta to discrete values, thereby replacing the integral in (3.24) by an infinite sum. In fact, the full spectrum of this circle theory is

$$\mathcal{H} = \bigoplus_{m,n} \mathcal{H}_{(m,n)}, \quad (3.35)$$

where  $\mathcal{H}_{(m,n)}$  consists of the states that are generated by the action of the negative modes  $\alpha_{-l}$  and  $\bar{\alpha}_{-l}$  with  $l > 0$  from a ground-state  $|(p_L, p_R)\rangle$  for which

$$\alpha_0 |(p_L, p_R)\rangle = p_L |(p_L, p_R)\rangle \quad \bar{\alpha}_0 |(p_L, p_R)\rangle = p_R |(p_L, p_R)\rangle, \quad (3.36)$$

with

$$(p_L, p_R) = \left( \frac{m}{R} + nR, \frac{m}{R} - nR \right). \quad (3.37)$$

Note in particular, that  $p_L \neq p_R$  in general; this is due to the possibility that the (closed) string may wind around the target space circle. Indeed, if we replace  $p$  by  $p_L$  and  $p_R$  in (3.2) and (3.3), respectively, then the expansion of  $X(\tau, \sigma)$  contains the terms

$$X(\tau, \sigma) = x + \frac{1}{2}(p_L + p_R)\tau + \frac{1}{2}(p_L - p_R)\sigma + \cdots . \quad (3.38)$$

For the above values of  $(p_L, p_R)$  this becomes

$$X(\tau, \sigma) = x + \frac{m}{R}\tau + nR\sigma , \quad (3.39)$$

and thus  $X(\tau, 2\pi) - X(\tau, 0) = 2n\pi R$ , describing a string that winds  $n$  times around the compact circle direction.

As before, the boundary states that preserve the current symmetry satisfy either a Neumann gluing condition (3.16) or a Dirichlet gluing condition (3.18). For  $l = 0$  (3.16) implies that a Neumann Ishibashi state can only be constructed in  $\mathcal{H}_{(m,n)}$  provided that  $p_L = -p_R$ . (In terms of our previous discussion this is simply the statement that the left- and right representations of the preserved symmetry algebra must be conjugate representations.) At a generic radius  $R$ ,  $p_L = -p_R$  can only be satisfied if  $m = 0$ , and thus we have a Neumann Ishibashi state for each  $n \in \mathbb{Z}$ ,

$$|(nR, -nR)\rangle\rangle^N \in \mathcal{H}_{(0,n)} . \quad (3.40)$$

Similarly, a Dirichlet Ishibashi state can only be constructed in  $\mathcal{H}_{(m,n)}$  provided that  $p_L = p_R$ ; at a generic radius we therefore only have the Dirichlet Ishibashi states

$$\left| \left( \frac{m}{R}, \frac{m}{R} \right) \right\rangle\rangle^D \in \mathcal{H}_{(m,0)} , \quad (3.41)$$

where  $m \in \mathbb{Z}$ . As before, one can easily give a closed formula for these Ishibashi states; they are simply given as

$$\begin{aligned} |(nR, -nR)\rangle\rangle^N &= \exp\left(\sum_{l=1}^{\infty} -\frac{1}{l} \alpha_{-l} \bar{\alpha}_{-l}\right) |(nR, -nR)\rangle \\ \left| \left( \frac{m}{R}, \frac{m}{R} \right) \right\rangle\rangle^D &= \exp\left(\sum_{l=1}^{\infty} \frac{1}{l} \alpha_{-l} \bar{\alpha}_{-l}\right) \left| \left( \frac{m}{R}, \frac{m}{R} \right) \right\rangle . \end{aligned} \quad (3.42)$$

The actual D-branes (that satisfy Cardy's condition) are given as linear combinations of these Ishibashi states. In the present case, the relevant expressions are

$$\|w\rangle\rangle = \frac{R^{\frac{1}{2}}}{2^{\frac{1}{4}}} \sum_{n \in \mathbb{Z}} e^{iwnR} |(nR, -nR)\rangle\rangle^N , \quad (3.43)$$

which describes a Neumann brane with Wilson line  $w$ , and

$$\|a\rangle\rangle = \frac{1}{2^{\frac{1}{4}} R^{\frac{1}{2}}} \sum_{m \in \mathbb{Z}} e^{i \frac{ma}{R}} \left| \left( \frac{m}{R}, \frac{m}{R} \right) \right\rangle\rangle^D, \quad (3.44)$$

which corresponds to a Dirichlet brane at the position  $a$ . Given the explicit form of the Ishibashi states, it is now straightforward to work out the closed string tree diagram, *i.e.* the overlap (3.25). For example, the overlap between two Dirichlet boundary states at the same position  $a$  is

$$\mathcal{A} = \langle\langle a \| e^{-tH_c} \| a \rangle\rangle = \frac{1}{\sqrt{2}R} \frac{1}{f_1(q)} \sum_{m \in \mathbb{Z}} \exp \left[ -\frac{t}{2} \pi \left( \frac{m}{R} \right)^2 \right]. \quad (3.45)$$

Using the Poisson resummation formula

$$\sum_{m \in \mathbb{Z}} \exp \left[ -\pi \frac{t}{2} \left( \frac{m}{R} \right)^2 \right] = \frac{\sqrt{2}R}{\sqrt{t}} \sum_{n \in \mathbb{Z}} \exp \left[ -2\pi \tilde{t} (Rn)^2 \right], \quad (3.46)$$

it follows that  $\mathcal{A}$  can be rewritten as

$$\mathcal{A} = \frac{1}{f_1(\tilde{q})} \sum_{n \in \mathbb{Z}} e^{-2\pi \tilde{t} (Rn)^2}, \quad (3.47)$$

which does indeed describe the correct open string partition function — the summation variable  $n$  labels the winding number of the open string. The discussion for the other boundary states is similar.

For both Neumann and Dirichlet branes one can also analyse the corresponding factorisation constraint. In both cases, the relevant classifying algebra simplifies considerably since the combination of  $C$  and  $F$  that appears in (2.38) is essentially trivial. More precisely, if we write the boundary states as

$$\begin{aligned} \|B^N\rangle\rangle &= \frac{R^{\frac{1}{2}}}{2^{\frac{1}{4}}} \sum_{n \in \mathbb{Z}} \hat{B}_n^N | (nR, -nR) \rangle\rangle^N \\ \|B^D\rangle\rangle &= \frac{1}{2^{\frac{1}{4}} R^{\frac{1}{2}}} \sum_{m \in \mathbb{Z}} \hat{B}_m^D \left| \left( \frac{m}{R}, \frac{m}{R} \right) \right\rangle\rangle^D, \end{aligned} \quad (3.48)$$

the factorisation constraint simply becomes

$$\begin{aligned} \hat{B}_{n_1}^N \cdot \hat{B}_{n_2}^N &= \hat{B}_{n_1+n_2}^N \\ \hat{B}_{m_1}^D \cdot \hat{B}_{m_2}^D &= \hat{B}_{m_1+m_2}^D. \end{aligned} \quad (3.49)$$

The most general fundamental U(1)-preserving Neumann and Dirichlet branes are thus described by

$$\hat{B}_n^N = e^{iwnR} \quad \hat{B}_m^D = e^{i\frac{ma}{R}}, \quad (3.50)$$

and therefore correspond to the branes given above. Strictly speaking, one could also choose  $w$  and  $a$  to be arbitrary complex (rather than real) numbers. While the resulting branes seem to be consistent from a conformal field theory point of view, they have complex couplings to some of the space-time fields, and are therefore presumably unphysical.

### 3.1.3 The conformal branes on the circle

The above circle theory is one of the few examples where one also knows how to describe the conformal branes, *i.e.* the branes that only satisfy (2.13), but not necessarily any of the current gluing conditions. The result depends crucially on the value of the radius  $R$ , namely on whether  $R$  is a rational or irrational multiple of the self-dual radius, which in the above conventions is  $R_c = 1$ . In the former case, *i.e.* if

$$R = \frac{M}{N}, \quad (3.51)$$

where  $M$  and  $N$  are coprime positive integers, the most general fundamental conformal D-branes can be described as follows [37]:<sup>8</sup> every fundamental conformal D-brane is (i) either a Neumann or Dirichlet brane (*i.e.* has a boundary state given by (3.43) or (3.44), respectively); or (ii) it is a brane associated to an element in

$$\mathrm{SU}(2)/\mathbb{Z}_M \times \mathbb{Z}_N. \quad (3.52)$$

If we write an arbitrary group element of SU(2) as

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad |a|^2 + |b|^2 = 1, \quad (3.53)$$

then the generator of  $\mathbb{Z}_N$  acts as  $a \mapsto e^{\frac{2\pi i}{N}} a$ , while the generator of  $\mathbb{Z}_M$  acts as  $b \mapsto e^{\frac{2\pi i}{M}} b$ . The branes associated to (3.52) are fundamental provided that  $ab \neq 0$ ; on the other hand, for  $a = 0$  the brane associated to (3.53) is the superposition of  $N$  Neumann branes (3.43) with evenly spaced Wilson lines, while for  $b = 0$ , the brane described by (3.53) is the superposition of  $M$  equidistantly spaced Dirichlet branes (3.44). The general D-branes in the family interpolate between these two extremal configurations. In fact, the Dirichlet

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<sup>8</sup>Here we mean by ‘fundamental’ simply that the open string with both ends on the same brane contains the vacuum representation with multiplicity one.

or Neumann brane configurations merge into intermediate boundary states that can no longer be thought of as superpositions of fundamental branes. These intermediate branes are themselves fundamental, and do not preserve the U(1) symmetry.

The situation at an irrational radius can be formally deduced from the above by taking simultaneously  $M, N \rightarrow \infty$ . In this limit the branes labelled by (3.52) then only depend on the modulus of  $a$  and  $b$ . Since  $|a|^2 + |b|^2 = 1$ , there is therefore only one real parameter that we can take to be given by  $x = 2|a|^2 - 1$  with  $-1 \leq x \leq 1$ . In addition to the standard Neumann and Dirichlet branes the theory therefore has only an interval of branes labelled by  $x$  [38, 39]. This interval of branes interpolates between a smeared Dirichlet brane (*i.e.* the integral of Dirichlet boundary states where we integrate over all possible positions on the circle) and a smeared Neumann brane (*i.e.* the integral of Neumann boundary states where we integrate over all possible Wilson lines on the dual circle).

## 3.2 Introducing fermions

Up to now we have only discussed bosonic conformal field or string theories. For (world-sheet) fermions a few additional complications arise. In the following we shall always discuss the NS-R formalism; the description of D-branes in the Green-Schwarz formalism is similar [47].

### 3.2.1 The spin structure

Let us consider the example of ten-dimensional superstring theory. The bosonic degrees of freedom (on the world-sheet) are described precisely as above, the only exception being that there are now only ten coordinate fields  $X^\mu$ , which give rise to eight transverse degrees of freedom in light-cone gauge. [We shall continue to work in light-cone gauge in the following.] In addition we now have eight left- and right-moving fermion fields of conformal weight  $h = 1/2$  and  $\bar{h} = 1/2$ ,

$$\psi^\mu(z) = \sum_r \psi_r^\mu z^{-r-1/2}, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_r \tilde{\psi}_r^\mu \bar{z}^{-r-1/2}. \quad (3.54)$$

Here  $r$  runs over all half-integers (integers) in the NS (R) sector. The anti-commutation relations of the modes are in all sectors given by

$$\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu} \delta_{r,-s}, \quad \{\psi_r^\mu, \tilde{\psi}_s^\nu\} = 0, \quad \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu} \delta_{r,-s}. \quad (3.55)$$

Let us consider again the D-branes that preserve the full current symmetry, and let us try to find corresponding boundary conditions for the free fermion fields. This is to say, let us impose the gluing conditions (2.12) also for the fermion fields. Since  $h = 1/2$ , the

prefactor  $(-1)^h$  that appears in the gluing condition (2.12) equals  $\pm i$ . For each fixed  $\rho$ , there are therefore two solutions that are parametrised by  $\eta = \pm$ , namely

$$\left(\psi_r^\mu + i\eta \rho(\tilde{\psi}_{-r}^\mu)\right) \|D, \eta\rangle\rangle = 0. \quad (3.56)$$

The boundary should not only preserve the free boson and free fermion symmetries separately, but also the superconformal symmetry of the world-sheet theory. The  $N = 1$  supercharge is of the form

$$G_r = \sum_{n \in \mathbb{Z}} \eta^{\mu\nu} \psi_{r-n}^\mu \alpha_n^\nu, \quad (3.57)$$

and similarly for  $\tilde{G}_r$ . In order for the boundary states to satisfy in addition

$$(G_r + i\eta \tilde{G}_{-r}) \|D, \eta\rangle\rangle = 0, \quad (3.58)$$

we need to choose the action of  $\rho$  in (3.56) to agree with that for the bosons, *i.e.* we have to choose  $\rho(\tilde{\psi}_{-r}^\mu) = +\rho(\tilde{\psi}_{-r}^\mu)$ , if  $\mu$  is a Neumann direction, and  $\rho(\tilde{\psi}_{-r}^\mu) = -\rho(\tilde{\psi}_{-r}^\mu)$ , if  $\mu$  is Dirichlet.

The possible Dirichlet branes of this type are therefore characterised by  $p$ , where  $p+1$  is the number of Neumann directions which we may again assume to be  $x^1, \dots, x^{p+1}$ , as well as by  $\eta = \pm$ . The relevant Ishibashi states can only exist in the NS-NS and R-R sector, *i.e.* in the sectors where both left- and right-movers are NS or both R, since otherwise (3.56) does not make any sense. Since this is again a free theory, we can simply write down these Ishibashi states

$$\begin{aligned} |Bp, \mathbf{k}, \eta\rangle\rangle = \exp \left\{ \sum_{n>0} \left[ -\frac{1}{n} \sum_{\mu=1}^{p+1} \alpha_{-n}^\mu \tilde{\alpha}_{-n}^\mu + \frac{1}{n} \sum_{\nu=p+2}^8 \alpha_{-n}^\nu \tilde{\alpha}_{-n}^\nu \right] \right. \\ \left. + i\eta \sum_{r>0} \left[ -\sum_{\mu=1}^{p+1} \psi_{-r}^\mu \tilde{\psi}_{-r}^\mu + \sum_{\mu=p+2}^8 \psi_{-r}^\mu \tilde{\psi}_{-r}^\mu \right] \right\} |\mathbf{k}, \eta\rangle^{(0)}, \quad (3.59) \end{aligned}$$

where the state  $|\mathbf{k}, \eta\rangle^{(0)}$  is simply the NS-NS ground state with momentum  $\mathbf{k}$  (where again  $k^\mu = 0$  for  $\mu = 1, \dots, p+1$ ), while in the R-R sector it is (uniquely) characterised by the fermionic gluing condition (3.56) with  $r = 0$ .

### 3.2.2 To GSO or not to GSO

If we were interested in constructing boundary conditions in conformal field theory, we would now go ahead and construct the boundary states for each choice of  $p$  and  $\eta$  separately. (This is to say, we would think of  $\eta$  as some part of the gluing automorphism  $\rho$ .) This can be done in close analogy to what was done above for the bosonic case.

However, in string theory, this is not quite what we are interested in. The boundary state should be an element of the closed string spectrum of the theory (*i.e.* it should only couple to physical states of the theory), but the actual closed string spectrum is not just the sum over the different sectors NS-NS, NS-R, R-NS and R-R. (Indeed, if this was the case, the theory would contain a tachyon, the ground state of the NS-NS sector, and would surely not be spacetime supersymmetric.)

As is well known, the actual spectrum of the closed string theory only consists of the states that are GSO-invariant. We therefore need to guarantee that the boundary states we are constructing are also GSO-invariant. As we shall see momentarily, this requires that we add together Ishibashi states with  $\eta = \pm$ .

In order to formulate the GSO-projection let us introduce the left- and right-moving fermion number generators,  $(-1)^F$  and  $(-1)^{\tilde{F}}$ . By construction,  $(-1)^F$  anti-commutes with left-moving fermionic modes, but commutes with all other modes, while  $(-1)^{\tilde{F}}$  anti-commutes with all right-moving fermionic modes, but commutes with all other modes. Furthermore, both have eigenvalue  $-1$  on the NS-NS ground state, and we choose some suitable convention on the R-R ground states. In the NS-NS sector we impose the GSO-projection

$$P_{\text{NS-NS}} = \frac{1}{4} \left( 1 + (-1)^F \right) \left( 1 + (-1)^{\tilde{F}} \right), \quad (3.60)$$

while in the R-R sector there are two different choices that correspond to type IIA and type IIB string theory. For IIA the relevant projector is

$$P_{\text{R-R}}^A = \frac{1}{4} \left( 1 + (-1)^F \right) \left( 1 - (-1)^{\tilde{F}} \right), \quad (3.61)$$

while for IIB it is

$$P_{\text{R-R}}^B = \frac{1}{4} \left( 1 + (-1)^F \right) \left( 1 + (-1)^{\tilde{F}} \right). \quad (3.62)$$

(There are also suitable projections in the NS-R and R-NS sectors, but they do not play a role for us in the following.)

Let us now discuss the effect of imposing the GSO-projection on the various Ishibashi states. In the NS-NS sector, it is easy to see (given that both  $(-1)^F$  and  $(-1)^{\tilde{F}}$  anti-commute with  $\psi_{-r}^\mu \tilde{\psi}_{-r}^\mu$ ) that the Ishibashi state (3.59) satisfies

$$(-1)^F |Bp, \mathbf{k}, \eta\rangle\rangle_{\text{NS-NS}} = (-1)^{\tilde{F}} |Bp, \mathbf{k}, \eta\rangle\rangle_{\text{NS-NS}} = -|Bp, \mathbf{k}, -\eta\rangle\rangle_{\text{NS-NS}}. \quad (3.63)$$

Thus the GSO-invariant Ishibashi state is therefore

$$|Bp, \mathbf{k}, \rangle\rangle_{\text{NS-NS}} = \frac{1}{\sqrt{2}} \left( |Bp, \mathbf{k}, +\rangle\rangle_{\text{NS-NS}} - |Bp, \mathbf{k}, -\rangle\rangle_{\text{NS-NS}} \right). \quad (3.64)$$

In the R-R sector, the analysis is somewhat more complicated, since there are fermionic zero modes that need to be taken into consideration. In order to describe this in detail, we need a little bit of notation. Let us define the modes

$$\psi_{\pm}^{\mu} = \frac{1}{\sqrt{2}} \left( \psi_0^{\mu} \pm i\tilde{\psi}_0^{\mu} \right), \quad (3.65)$$

which satisfy the anti-commutation relations

$$\{\psi_{\pm}^{\mu}, \psi_{\pm}^{\nu}\} = 0, \quad \{\psi_{+}^{\mu}, \psi_{-}^{\nu}\} = \delta^{\mu\nu}, \quad (3.66)$$

as follows from (3.55). [We are restricting our attention here to the ‘transverse’ directions for which  $\eta^{\mu\nu} = \delta^{\mu\nu}$ .] In terms of  $\psi_{\pm}^{\mu}$ , the zero mode condition of (3.56) can be written as

$$\begin{aligned} \psi_{\eta}^{\mu} |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} &= 0 & \mu &= 1, \dots, p+1 \\ \psi_{-\eta}^{\nu} |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} &= 0 & \nu &= p+2, \dots, 8. \end{aligned} \quad (3.67)$$

Because of the anti-commutation relations (3.66) we can define

$$|Bp, \mathbf{k}, +\rangle_{\text{R-R}}^{(0)} = \prod_{\mu=1}^{p+1} \psi_{+}^{\mu} \prod_{\nu=p+2}^8 \psi_{-}^{\nu} |Bp, \mathbf{k}, -\rangle_{\text{R-R}}^{(0)}, \quad (3.68)$$

and then it follows that

$$|Bp, \mathbf{k}, -\rangle_{\text{R-R}}^{(0)} = \prod_{\mu=1}^{p+1} \psi_{-}^{\mu} \prod_{\nu=p+2}^8 \psi_{+}^{\nu} |Bp, \mathbf{k}, +\rangle_{\text{R-R}}^{(0)}. \quad (3.69)$$

On the ground states the GSO-operators take the form

$$(-1)^F = \prod_{\mu=1}^8 (\sqrt{2}\psi_0^{\mu}) = \prod_{\mu=1}^8 (\psi_{+}^{\mu} + \psi_{-}^{\mu}), \quad (3.70)$$

and

$$(-1)^{\tilde{F}} = \prod_{\mu=1}^8 (\sqrt{2}\tilde{\psi}_0^{\mu}) = \prod_{\mu=1}^8 (\psi_{+}^{\mu} - \psi_{-}^{\mu}). \quad (3.71)$$

Taking these equations together we then find that

$$(-1)^F |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} = |Bp, \mathbf{k}, -\eta\rangle_{\text{R-R}}^{(0)} \quad (3.72)$$

$$(-1)^{\tilde{F}} |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} = (-1)^{p+1} |Bp, \mathbf{k}, -\eta\rangle_{\text{R-R}}^{(0)}. \quad (3.73)$$

The action on the non-zero modes is as before, and therefore the action of the GSO-operators on the whole Ishibashi states is given by

$$(-1)^F |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}} = |Bp, \mathbf{k}, -\eta\rangle_{\text{R-R}} \quad (3.74)$$

$$(-1)^{\tilde{F}} |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}} = (-1)^{p+1} |Bp, \mathbf{k}, -\eta\rangle_{\text{R-R}}. \quad (3.75)$$

It follows from the first equation that the only potentially GSO-invariant Ishibashi state is of the form

$$|Bp, \mathbf{k}\rangle_{\text{R-R}} = \frac{1}{\sqrt{2}} \left( |Bp, \mathbf{k}, +\rangle_{\text{R-R}} + |Bp, \mathbf{a}, -\rangle_{\text{R-R}} \right), \quad (3.76)$$

and the second equation implies that it is actually GSO-invariant if

$$p \text{ is } \begin{cases} \text{even} & \text{for IIA} \\ \text{odd} & \text{for IIB.} \end{cases} \quad (3.77)$$

To each such GSO-invariant Ishibashi states (3.64) and (3.76) we can then also construct the corresponding ‘boundary state’ by Fourier transformation,

$$\|Bp, \mathbf{a}\rangle = \mathcal{N} \int \prod_{\nu=p+2, \dots, 8} dk^\nu e^{ik^\nu a^\nu} |Bp, \mathbf{k}\rangle, \quad (3.78)$$

where  $\mathcal{N}$  is a suitable normalisation constant that equals

$$\mathcal{N}_{\text{NS-NS}} = \frac{1}{4} \quad \mathcal{N}_{\text{R-R}} = 1. \quad (3.79)$$

With this normalisation, the overlaps between these boundary states lead to

$$\langle\langle Bp, \mathbf{a}_1 \| e^{-tH_c} \| Bp, \mathbf{a}_2 \rangle\rangle_{\text{NS-NS}} = (2\tilde{t})^{-\frac{p+1}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2}{2\pi} \tilde{t}} \left( \frac{f_3^8(\tilde{q})}{f_1^8(\tilde{q})} - \frac{f_2^8(\tilde{q})}{f_1^8(\tilde{q})} \right), \quad (3.80)$$

as well as

$$\langle\langle Bp, \mathbf{a}_1 \| e^{-tH_c} \| Bp, \mathbf{a}_2 \rangle\rangle_{\text{R-R}} = (2\tilde{t})^{-\frac{p+1}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2}{2\pi} \tilde{t}} \frac{f_4^8(\tilde{q})}{f_1^8(\tilde{q})}. \quad (3.81)$$

### 3.2.3 Stable BPS branes

We are now in the position to discuss briefly the various types of D-branes that exist in either of these theories. First of all, the stable BPS D-branes are given by the boundary states

$$\|Dp\rangle = \frac{1}{\sqrt{2}} \left( \|Dp\rangle_{\text{NS-NS}} \pm i \|Dp\rangle_{\text{R-R}} \right), \quad (3.82)$$

where the sign between the NS-NS and the R-R component in (3.82) distinguishes between a brane and an anti-brane. These boundary states involve a R-R component, and they are therefore only GSO-invariant provided that  $p$  satisfies (3.77). The D-branes satisfy the appropriate form of the Cardy condition, in that they lead to open strings of the form

$$[\text{NS} - \text{R}] \frac{1}{2} (1 + (-1)^F). \quad (3.83)$$

The GSO-projection in the open string removes the tachyon from the open string NS sector, and the branes are therefore stable. One can show that these branes are actually BPS, *i.e.* that they are annihilated by one half of the space-time supercharges. [The spacetime supercharges map the NS-NS into the R-R sector, and thus the presence of the R-R sector is also required for supersymmetry.]

### 3.2.4 Unstable non-BPS branes

The theory also possesses unstable *non-BPS* D-branes [10] whose boundary states are of the form [49]

$$\|Dp\rangle\rangle = \|Dp\rangle\rangle_{\text{NS-NS}}. \quad (3.84)$$

As is explained in detail in [50], these branes occur for the complementary values of  $p$  relative to (3.77). Their open string spectrum is now of the form

$$[\text{NS} - \text{R}] \quad (3.85)$$

without a GSO-projection. As a consequence the open string tachyon from the NS-sector is not removed, and the D-brane is unstable.

### 3.2.5 Stable non-BPS branes

As we have seen above, all stable (fundamental) U(1)-preserving D-branes of Type IIA/IIB string theory in flat space are actually BPS. However, this is not true in general. In particular, stable non-BPS D-branes exist for certain orbifold theories of Type IIA/IIB. The simplest example is the D0-brane of the orbifold of IIB on  $T^4/(-1)^{F_L}\mathcal{I}_4$  [51, 52]. Here  $\mathcal{I}_4$  denotes the inversion of the four torus directions, and  $(-1)^{F_L}$  acts as  $\pm 1$  on left-moving space-time fermions. The corresponding boundary state is schematically of the form

$$\|D0\rangle\rangle = \frac{1}{\sqrt{2}} \left( \|D0\rangle\rangle_{\text{NS-NS}} \pm i \|D0\rangle\rangle_{\text{R-R;T}} \right), \quad (3.86)$$

where the second component is the appropriately normalised (GSO-invariant) Ishibashi state coming from the twisted R-R sector. This boundary state is not BPS since it does

not have a component in the untwisted R-R sector. However, it is nevertheless stable (at least for sufficiently large radii) since the corresponding open string is

$$[\text{NS} - \text{R}] \frac{1}{2} (1 + (-1)^F I_4). \quad (3.87)$$

This projection removes the zero-winding component of the tachyon in the open string NS-sector, and thus stabilizes the D-brane.

Stable non-BPS D-branes play an important role for understanding string dualities of supersymmetric string theories. For example the T-dual of the above D0-brane is a non-BPS D1-brane for IIA on  $T^4/\mathcal{I}_4$  [52, 11]. Since  $T^4/\mathcal{I}_4$  is an orbifold limit of K3, this theory is dual to the Heterotic string on  $T^4$ . The dual of the non-BPS D1-brane of the IIA theory can then be identified with a certain perturbative stable non-BPS state of the Heterotic theory [11]. This sheds some light on how string duality relates states that are not BPS, and that are therefore not protected from quantum corrections.

### 3.3 K-theory charges

The above non-BPS D-branes can be understood to arise from suitable brane – anti-brane configurations via tachyon condensation [53, 52, 54, 55, 56, 10, 57]. For example, the non-BPS D0-brane of the above orbifold (section 3.2.5) can be understood to arise from a D1-brane – anti-D1-brane pair that wraps around one of the four circles that are inverted under the action of  $\mathcal{I}_4$  [52]. Because of the orbifold symmetry, only two discrete values for the Wilson line are allowed, and the brane and the anti-brane are taken to have different Wilson lines. The total brane – anti-brane configuration then only carries twisted R-R charge at one fixed point, and thus has the same R-R charges as the non-BPS D0-brane. Depending on the radius of the circle, only one of the two configurations (*i.e.* the brane – anti-brane configuration or the non-BPS D0-brane) is stable, and at the critical radius, they are related by a marginal deformation.

Similarly, the unstable non-BPS D-branes of section 3.2.4 can be obtained from brane anti-brane pairs. Consider, for example, the configuration in IIB string theory of a BPS D1-brane together with a coincident anti-D1-brane. This system contains a complex tachyon field on the world-volume of the brane, whose potential is described by figure 1<sup>9</sup>. At  $T = 0$ , the system is at the maximum of the tachyon potential and therefore unstable. On the other hand, at the ground state of the tachyon potential (*i.e.* for  $|T| = T_0$ ), the brane and anti-brane have completely disappeared, and the configuration is indistinguishable from the vacuum. [In particular, this requires that  $V(T_0) = -2T_D$ , where  $T_D$  is the tension of the relevant D-brane [54]. This relation

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<sup>9</sup>This figure has been taken from [57].

Figure 2: The tachyon potential on the brane - anti-brane pair.

was subsequently checked to high accuracy in string field theory — see [58] for a recent account of the status of these calculations.]

Now the idea of the construction is that the tachyon need not take the *same* expectation value everywhere on the world-volume of the brane. For example, if the world-volume of the D1-brane spans  $x^0$  and  $x^1$ , we could choose  $T$  to be a real (say odd) function of  $x^1$  such that

$$\begin{aligned} T(x^1) &\rightarrow -T_0 && \text{as } x^1 \rightarrow -\infty \\ T(x^1) &\rightarrow T_0 && \text{as } x^1 \rightarrow \infty. \end{aligned}$$

The idea is that this ‘kink’-profile describes the unstable D0-brane (at  $x^1 = 0$ ). This makes sense since for large  $|x^1|$ , the above configuration is indistinguishable from the vacuum, and only near  $x^1 = 0$  do we expect some remnant of the original D-branes. Clearly this kink is unstable since it can be pulled off the top of the potential, but this just corresponds to the fact that the D0-brane is in fact unstable in IIB string theory.

The above construction can actually be generalised to all D-branes, not just non-BPS branes. For example, in the above configuration, there is a stable ‘vortex solution’ where  $T$  is a function of  $x^0$  and  $x^1$  for which

$$|T(\mathbf{x})| \rightarrow T_0 \quad \text{as } |\mathbf{x}| \rightarrow \infty, \tag{3.88}$$

and  $\mathbf{x} = (x^0, x^1)$ . This solution is stable, *i.e.* it cannot be deformed with finite energy, provided that it has a non-trivial winding number  $w$ . [Here, the winding number  $w$  is the

number of times  $T(\mathbf{x})$  (for  $|\mathbf{x}|$  large) winds around the circle  $|T| = T_0$  as  $\mathbf{x}$  circles once around the origin in the plane.] If  $w \neq 0$ , then this solution describes  $w$  D-instantons. (As before, because the tachyon is at the minimum of its potential for all large  $|\mathbf{x}|$ , the resulting D-brane must have co-dimension two, and therefore must be the point-like D-instanton.)

Obviously this construction can be repeated for any D-brane, and one can therefore, successively, construct any of the above D-branes starting from a suitable number of D9-brane – anti-D9-brane pairs (for the case of IIB string theory — there are corresponding constructions for the other cases). One can think of the tachyon field as a map between the vector bundle of the  $N$  D9-branes, and the vector bundle of the  $N$  anti-D9-branes, and thus recast the above construction in terms of K-theory [5] (see also [49, 59]). Thus the charges that are carried by the D-branes can be interpreted as ‘topological’ K-theory charges.

In general one therefore expects that the D-brane charges (that can be calculated in terms of conformal field theory) should agree with the K-theory charges of the corresponding target space, and in many examples these two descriptions do indeed agree (see for example [5, 49, 59, 15]). As far as I am aware, there is currently only one class of examples, namely the WZW models, for which this has not been satisfactorily understood: the D-brane charges of the branes that preserve the full affine symmetry have been determined in [60, 61, 62], but they do not, in general, account for all the K-theory charges that have been found [63]. One expects that the remaining charges correspond to D-branes that do not preserve the full current algebra, but a detailed conformal field theory description of these branes is currently still lacking. In particular, it seems that the D-branes that preserve the affine symmetry up to an outer automorphism are not sufficient to account for all the missing charges.

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